UNIVERSITÀ DEGLI STUDI DI CAGLIARI
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Simmetrie asintotiche dello spazio anti-de Sitter in due e tre dimensioni

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Asymptotic symmetries of anti-de Sitter space in two and three dimensions

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Translated into English by Luca Mana.

I dedicate this English version
to my grandmother, nonna Tilde,
whose presence and love I miss a lot.

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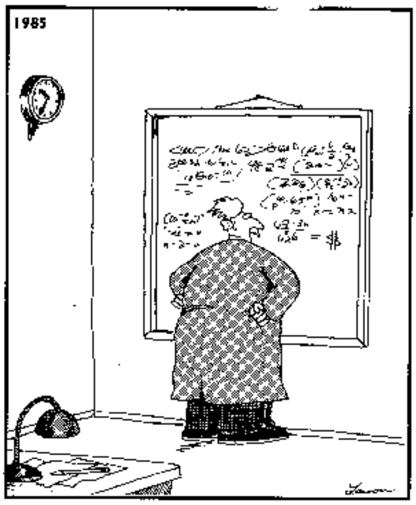

Einstein discovers that time is actually money.

Introduction

The context

It is no easy task to unravel the rather interwoven context wherein the present work finds its place into a sequential thread; and, just as a one-to-one mapping from the plane onto the line necessarily breaks the original topology, so a sequential exposition can but show concepts that are near as they were far apart. Nevertheless we shall try to describe this context and give a frame to the motivations, the goals, the results of this work. The Grand Unification of the fundamental forces of nature, String Theory, The Holographic Principle, and General Relativity are the four pillars whereupon this context rests (q.v. Polchinski [38]).

The Grand Unification and String Theory

The quest for a unitary description of nature[1] by means of one fundamental theory has marked theoretical physics in this century, and very likely will mark it in the coming century as well.

Two main questions — still unanswered — in this quest are the consistent quantization of the gravitational field and the unification of the four fundamental interactions at high energies. String Theory appears to be the most promising theory to solve these questions today.

The key idea in String Theory is that particles, i.e. the fields' quanta, should be unidimensional, rather than point-like, objects; though their unidimensionality should manifest itself at a very microscopical length scale — Planck's scale, $L_P \sim 10^{-33}$ cm — and so at very high energies, $\sim 10^{16}$ GeV. This simple idea leads to the solution of the 'ultraviolet plague' and to the unification of the four fundamental forces; but String Theory has other good features as well: at low energies it yields General Relativity as an effective theory; it incorporates most fashionable useful physical-mathematical concepts such as supersimmetry and symmetry breaking; its mathematical form is such that many important parameters (e.g. spacetime dimensionality) are uniquely determined just by requiring mathematical consistency.

[1] In fact, a number of evidences seems to show that such a description could be non-unitary instead (in the quantum-mechanical sense).

Yet String Theory is very complex mathematically — just because it replaces point-like with unidimensional objects —, and this has been the main obstacle against its full acceptance so far; indeed, one cannot make many univocal predictions at the energy scales which one can probe by today's particle accelerators.

However, recent studies emphasize the importance of the theory's non-perturbative structure, where one finds solutions like D-branes and black holes, and a recent principle is proving to be very useful in this respect: the *Holographic Principle*, which seems to be capable of yielding new results from the theory in regimes that cannot be perturbatively analysed.

The Holographic Principle

The adjective 'holographic' refers to the coding of a system's *whole* information onto a part of the system itself, e.g. its surface, in analogy with holography, where three-dimensional optical information is coded on a (bidimensional) surface.

The Holographic Principle was formulated for the first time by 't Hooft [46], who showed how different kinds of string theories correspond, or are dual, to different kinds of gauge theories in the limit-case where these have an infinite number of colour charges. This fact is the source of many important consequences. First, the duality is, in certain cases, between different energetic and coupling regimes, so that non-perturbative results for a string theory may correspond to perturbative ones for the dual gauge theory, and vice-versa. Then, since at low energies a string theory reduces to a gravity theory (possibly with additional fields, like e.g. the dilaton field), one can find remnants of the duality at an effective-theory level.

A more recent and stronger formulation of the Holographic Principle is due to Susskind [43], who states that a gravity theory on (four-dimensional) spacetime is equivalent to a non-gravitational theory on a (two-dimensional) surface, with an upper bound for the information(the *Holographic Bound*) of 1 bit per Planck area (${L_P}^2$); this bound comes from the fact that infrared effects in the bulk theory correspond to ultraviolet effects in the surface theory (q.v. Susskind and Witten [44]).

When the coupling constant of the dual gauge theory does not depend on the energy scale, then the dual gauge theory is conformally invariant. Since the conformal group in D dimensions is isomorphic to $SO(D,2)$[2], the dual string theory must contain this symmetry group; this in turn implies that the manifold whereupon the string theory lives must be $\mathrm{AdS}_{D+1} \times S^{D+1}$, where AdS_n is n-dimensional anti-de Sitter space. Following the principle further, a great number of colour charges corresponds to low energy regimes of the string theory, which thus reduces to a gravity theory on (D+1)-dimensional anti-de Sitter space. So, in this case, the Holographic Principle appears as a duality principle between D-dimensional conformal field theory and gravity theory on (D+1)-dimensional

[2]Except for $D = 1$ and $D = 2$, where it is infinite-dimensional.

anti-de Sitter space, as stated by Maldacena [35], and many interesting results and interesting interpretations of known results emerge from this duality. As an example, for $D = 4$ the principle puts into correspondence non-perturbative Yang-Mills theory with low-energy gravity theory, so that by studying the latter one obtains results concerning the former. For $D \leq 2$ the opposite situation happens: conformal symmetry becomes infinite-dimensional and one can study the conformal field theory in order to solve problems of the gravity theory; indeed one has a glimpse of a possible *statistical* interpretation for the entropy of a three-dimensional black hole.

Gravity theory on $(D+1)$-dimensional anti-de Sitter space has got one more dimension than the conformal dual theory, which lives on D-dimensional Minkowski space (short of topological identifications). Since the boundary of anti-de Sitter space is conformal to Minkowski space, one finds it natural to suppose that the dual theory should be some way related to this boundary. An exact correspondence between the two theories is not at hand yet, but remarkable improvements have been made thanks to the identification of the two theories' symmetries: since the dual conformal theory is related to the boundary of anti-de Sitter space at infinity, its symmetries are identified with the *asymptotic symmetries* of anti-de Sitter space.

Asymptotic symmetries are the symmetries that a gravity theory possesses at infinity. They are studied by specifying suitable *asymptotic conditions* for the fields of the theory. Charges are associated to the asymptotic symmetries: usual charges like mass or angular momentum, but also other charges which make the algebra of the symmetry generators highly untrivial. The most famous case is three-dimensional anti-de Sitter space, and was studied by Brown and Henneaux [14] at the end of the eighties; in this case the asymptotic symmetries constitute an infinite-dimensional group whose generators forms two copies of a Virasoro algebra with a definite central charge: this group is actually the group of conformal transformations in two dimensions. Having a correspondence between the symmetry groups of the two theories, one can go on to suppose a correspondence between states, and can count the microstates in the conformal theory corresponding to a given macrostate which is represented by a black hole in the gravity theory. Thus one can calculate, by statistical means, black-hole entropy — a quantity which had always been computable by semiclassical thermodynamic means only.

The Holographic Principle could be also a way to work out the so-called *information paradox*, consisting in the fact that a black hole, eventually, evaporates completely, according to semiclassical predictions, and thus all information that has been trapped inside its surface during its formation process gets lost. The paradox might be solved, for the information would not really be trapped inside the event horizon, but rather would be found coded on the boundary.

Thus one can see how the Holographic Principle should some way manifest itself even at a semiclassical or classical level, within General Relativity.

General Relativity

It is a general characteristic of gravity theories the possibility of coding all the system physical information, or part thereof, on a surface; this possibility is mathematically shown by the fact that the system's charges appear as surface integrals. This is due to the diffeomorphism-invariance of the theory: a kind of invariance which introduces many unphysical, gauge degrees of freedom. The idea here is that all effective, physical degrees of freedom can be found on the system boundary.

The development of a Hamiltonian formalism for gravity theory has helped to shed light on this point. In the Hamiltonian formalism, indeed, it is of great importance the distinction between the system's phase-space coordinates on the one hand, and the time coordinate, which marks the system's evolution and dynamics, on the other hand; or, in short, the distinction between space and time. As soon as a gravity theory is formulated in a Hamiltonian form, a profound difference is set up between spatial coordinates and the temporal one: diffeomorphism-invariance almost disappears, but at the same time almost all unphysical degrees of freedom disappear as well.

An interesting feature of the gravitational Hamiltonian, as opposed to the Hamiltonians of other theories, is that it needs a boundary integral to be well-defined, as was clearly shown by Regge and Teitelboim [39]. More recently, York [47] demonstrated that the gravitational Lagrangian needs additive boundary terms as well, in order to yield a better defined variational principle. So a new interest has flourished, in these years, just in those surface terms that university students learn to discard after applying Stokes' Theorem in the variational calculation. The Hamiltonian boundary terms yield the system's (conserved) charges, and, following Brown and York's quasilocal formalism [17], the Lagrangian boundary terms lead to (conserved) quantities as well.

The exact form of the boundary integral is still an object of research and discussion in the literature, though; special problems remain in finding a general expression for that part of the integral, the so-called 'counterterm', which allows one to obtain renormalised results when the system's boundary is pushed to infinity. A way for its construction has long been adopted, that refers to a background spacetime or 'ground state'; more recently another, 'intrinsic' way, which refers to the boundary geometrical objects, has been proposed [5]. Both these methods rest on reasonable theoretical grounds, but some ambiguities make them unsteady; anyway, it is clear that they are more similar to each other than it may seem, and that they are asymptotically equivalent [33].

The way toward a general exact expression for the boundary terms lies upon a deeper understanding of the relationship between the ground state and the excited states (say, black-hole states) of the theory. From the point of view of the Holographic Principle, finding a general correct expression is important for correctly obtaining the charges associated to the asymptotic symmetries and, hence, to the dual theory.

The present work

In the previously outlined context, the present work moves along the three directions that follow.

The suitable asymptotic conditions for a three-dimensional dilatonic theory (of the Jackiw-Teitelboim kind) on anti-de Sitter space will be studied; thence the asymptotic symmetries and the charges will be obtained. A comparison with the corresponding three-dimensional non-dilatonic theory will be made, examining how the dilaton's presence breaks the symmetries: we shall find that the symmetries form a finite-dimensional group (the ground-state isometry group), unlike the non-dilatonic case where the group has infinite dimensions; but the presence of the dilaton leads to diverging charges, forcing the group to be even smaller in order to avoid them.

A discussion will be made about the way Brown and York's formalism can be applied to the study of asymptotic symmetries and the calculation of their associated charges; some explicit calculations in anti-de Sitter space (for the three- and two-dimensional dilatonic cases, and the three-dimensional non-dilatonic case) will be given as examples, and a comparison with the already known results found through the Hamiltonian method will be made. We shall see how Brown and York's formalism is not completely fit for studying the asymptotic symmetries, neither are some Hamiltonian boundary terms recently proposed in the literature.

Moreover, in the calculations for the two-dimensional case, we shall use both the background reference counterterm and the intrinsic counterterm, comparing them, and pointing out the ambiguities which affect the latter in the presence of a dilatonic field.

Structure of the discussion

The present work is structured into four chapters.

In the first chapter, the main concepts and objects which will serve for the subsequent calculations are defined: the manifold which hosts the metric and dilaton fields, its boundary, the Lagrangian and Hamiltonian formulations of gravity theory, anti-de Sitter space and black-hole solutions.

The concepts of asymptotic condition, asymptotic symmetry, and associated charge are defined in the second chapter, and the Hamiltonian and Brown and York's methods for computing the charges are outlined; a discussion about the generalisation of the latter method to a dilatonic theory is made. Some of the boundary terms proposed so far in the literature are examined and compared.

In the third chapter explicit calculations of the asymptotic conditions and symmetries and charges for gravity on two- and three-dimensional anti-de Sitter space, with and without dilaton field, are made; first through Hamiltonian formalism and then through Brown and York's formalism; both background and intrinsic counterterms are used. The statistical method for entropy calculation through the conformal dual theory is sketched.

Finally, final remarks and conclusions are left to the fourth chapter.

Chapter 1

Generalia

1.1 Definitions

1.1.1 Notation

The following typographical conventions will be used to distinguish among different geometrical objects: scalars and tensors will be in italic (e.g. η, $T^{\mu\nu}$), scalar and tensor densities in boldface italic (e.g. $\boldsymbol{P}^{ab}$, $\boldsymbol{\Pi}^{\eta}$, $\sqrt{-\boldsymbol{g}}$, $\sqrt{\boldsymbol{h}}$), operators in boldface Roman (e.g. $\mathbf{\nabla}$, $\mathbf{D}$, $\mathbf{L}$), manifolds in calligraphic style (e.g. $\mathcal{M}$, $\mathcal{B}$), integrals in Fraktur (e.g. $\mathfrak{L}$, $\mathfrak{J}$); D-dimensional Minkowski space is $\mathbb{M}^D$, and finally see next section for convention on tensor indices.

Integrals will not show coordinates ($d^D x$, etc.) — easily inferable from the integration manifold, which will always be indicated —, except in cases of non-generic dimensionality.

The natural system of units ($c = G = \hbar = 1$) will be used throughout.

1.1.2 Main geometrical objects

We shall work on a $(D+1)$-dimensional differentiable manifold $\mathcal{M}$, whose boundary $\partial\mathcal{M}$ is given by the union of the D-dimensional hypersurfaces $\mathcal{S}'$, $\mathcal{S}''$, and $\mathcal{B}$; the first and the second are homeomorphic to the interior of S^D and the third to $S^{D-1} \times I$, where I is a real interval. The intersection between $\mathcal{S}'$ and $\mathcal{B}$ is the $(D-1)$-dimensional surface $\mathcal{P}'$, which is the boundary of $\mathcal{S}'$; an analogous definition holds for $\mathcal{P}''$; $\mathcal{P}'$ and $\mathcal{P}''$ constitute the (disconnected) boundary of $\mathcal{B}$.

The surfaces $\mathcal{S}'$ and $\mathcal{S}''$ can be thought of as the initial and final surfaces of a foliation of $\mathcal{M}$ into hypersurfaces $\mathcal{S}_t$ (or $\mathcal{S}$ for short). This foliation induces a foliation on $\mathcal{B}$ into surfaces $\mathcal{P}_t$ (or $\mathcal{P}$ for short), $\mathcal{P}_t \equiv \mathcal{B} \cap \mathcal{S}_t$, with $\mathcal{P}'$ and $\mathcal{P}''$ as extrema.

One can choose an adapted coordinate system on $\mathcal{M}$,

$$\{x^\mu\} \equiv \{x^0, x^i\} \equiv \{x^a, x^D\} \equiv \{x^0, x^A, x^D\} \equiv \{t, x^A, r\}, \tag{1.1}$$

with the following conventions for indices:

$$\mu, \nu, \text{ etc.} \in \{0, \dots, D\}, \quad (1.2a)$$
$$i, j, \text{ etc.} \in \{1, \dots, D\}, \quad (1.2b)$$
$$a, b, \text{ etc.} \in \{0, \dots, D-1\}, \quad (1.2c)$$
$$A, B, \text{ etc.} \in \{1, \dots, D-1\}, \quad (1.2d)$$

this coordinate system is adapted to the foliation so that the $\mathcal{S}$ hypersurfaces are given by $t =$ const., the boundary $\mathcal{B}$ by $r =$ const., and the $\mathcal{P}$ surfaces by $t, r =$ const. It is always possible.to choose such a coordinate system locally.

The manifold $\mathcal{M}$ is given a pseudo-Riemannian metric $g_{\mu\nu}$ (signature $(-, +, \dots, +)$), with connexion $\boldsymbol{\nabla}$ and scalar curvature $R_{\mathcal{M}}$. This metric structure induces other metric structures on the various surfaces (Fig. 1.1):

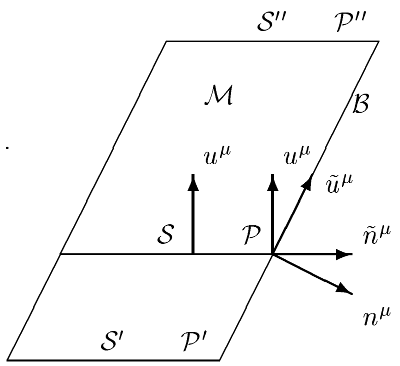

Figure 1.1: Example of a foliation for $D = 1$ (in this case the $\mathcal{P}$ surfaces degenerate into pair of points); the different normal vectors are shown.

- the $\mathcal{S}$ surfaces are spacelike with future-oriented, timelike unit normal vector field u^μ; they have induced metric $h_{\mu\nu} \equiv g_{\mu\nu} + u_\mu u_\mu$, linear connexion $\mathbf{D}$, intrinsic curvature $R_{\mathcal{S}}$, and extrinsic curvature $K_{\mu\nu} \equiv -\frac{1}{2}\mathbf{L}_u h_{\mu\nu} \equiv -h_\mu{}^\sigma h_{\nu\tau} \boldsymbol{\nabla}_\sigma u^\tau$; two projection operators are given: $h^\mu{}_\nu$ projects a tensor index onto $\mathcal{S}$, while $-u^\mu u_\nu$ projects onto the normal;
- the hypersurface $\mathcal{B}$ is timelike with outward-pointing, spacelike unit normal vector field n^μ; the induced metric is $\gamma_{\mu\nu} \equiv g_{\mu\nu} - n_\mu n_\mu$, with connexion $\boldsymbol{\Delta}$ and extrinsic curvature $\Theta_{\mu\nu} \equiv -\gamma_\mu{}^\sigma \gamma_{\nu\tau} \boldsymbol{\Delta}_\sigma n^\tau$; $\gamma^\mu{}_\nu$ projects a tensor index onto $\mathcal{B}$, while normal projection is done by $n^\mu n_\nu$; the hyperbolic angle between the normal vector fields u^μ and n^μ is $\alpha \stackrel{\text{def}}{=} -\operatorname{arcsinh} u^\mu n_\mu$;
- every surface $\mathcal{P}$ is spacelike with induced metric $\sigma_{\mu\nu}$ ($\sigma^\mu{}_\nu$ operates tangential projection), and has four unit normal vector fields:

1. u^μ, as it is a submanifold of $\mathcal{S}$;
2. n^μ, as it is a submanifold of $\mathcal{B}$;
3. $\tilde{n}^\mu$, an outward-pointing, spacelike unit vector field which is normal to $\mathcal{P}$ with respect to its embedding in $\mathcal{S}$;
4. $\tilde{u}^\mu$, a future-pointing, timelike unit vector field which is normal to $\mathcal{P}$ with respect to its embedding in $\mathcal{B}$.

The components of the metric objects above have simple expressions in the adapted coordinate system (the usual Arnowitt-Deser-Misner [3] decomposition):

$$(g_{\mu\nu}) \equiv \begin{pmatrix} -N^2 + N^k N_k & N_j \\ N_i & h_{ij} \end{pmatrix} \tag{1.3a}$$

$$\equiv \begin{pmatrix} \gamma_{ab} & V_b \\ V_a & V^2 + V^c V_c \end{pmatrix}, \tag{1.3b}$$

$$(g^{\mu\nu}) \equiv \begin{pmatrix} -\dfrac{1}{N^2} & \dfrac{N^j}{N^2} \\ \dfrac{N^i}{N^2} & h^{ij} - \dfrac{N^i N^j}{N^2} \end{pmatrix} \tag{1.3c}$$

$$\equiv \begin{pmatrix} \gamma^{ab} + \dfrac{V^a V^b}{V^2} & -\dfrac{V^b}{V^2} \\ -\dfrac{V^a}{V^2} & \dfrac{1}{V^2} \end{pmatrix}, \tag{1.3d}$$

where

$$h^{ik} h_{kj} \equiv \delta^i{}_j, \tag{1.4a}$$

$$\gamma^{ac}\gamma_{cb} \equiv \delta^a{}_b, \tag{1.4b}$$

$$N_i \stackrel{\text{def}}{=} N^k h_{ki}, \tag{1.4c}$$

$$V_a \stackrel{\text{def}}{=} V^c \gamma_{ca}; \tag{1.4d}$$

(so that h^{ij} and γ^{ab} are the inverse metrics in $\mathcal{S}$ and $\mathcal{B}$ respectively). N and N^i are the lapse and shift (with $N^0 \equiv N^t = 0$ and $V^D \equiv V^r = 0$ by definition). Moreover one has:

$$(u^\mu) \equiv \left(\frac{1}{N}, -\frac{N^i}{N}\right), \tag{1.5a}$$

$$(n^\mu) \equiv \left(-\frac{V^a}{V}, \frac{1}{V}\right), \tag{1.5b}$$

$$(u_\mu) \equiv (-N, \vec{0}), \tag{1.5c}$$

$$(n^\mu) \equiv (\vec{0}, V). \tag{1.5d}$$

Since the boundary $\mathcal{B}$ is foliated as well, its metric can be decomposed as

$$(\gamma_{ab}) \equiv \begin{pmatrix} -\tilde{N}^2 + \tilde{N}^C \tilde{N}_C & \tilde{N}_B \\ \tilde{N}_A & \sigma_{AB} \end{pmatrix}, \tag{1.6}$$

where $\tilde{N}$ and $\tilde{N}^A$ are the *boundary* lapse and shift, with $\tilde{N}^0 \equiv \tilde{N}^t = \tilde{N}^D \equiv \tilde{N}^r \stackrel{\text{def}}{=} 0$.

When the $\mathcal{S}$ hypersurfaces are orthogonal to $\mathcal{B}$ (i.e. when $\alpha = 0$), one has:

$$\tilde{u}^\mu = u^\mu, \tag{1.7a}$$

$$\tilde{n}^\mu = n^\mu, \tag{1.7b}$$

$$N = \tilde{N}, \tag{1.7c}$$

$$N^A|_{\mathcal{P}} = \tilde{N}^A, \tag{1.7d}$$

$$N^r|_{\mathcal{P}} = 0. \tag{1.7e}$$

Finally, one can consider the asymptotic limit $r \to \infty$ where the $\mathcal{S}$ hypersurfaces assume a spatially infinite extension, $\mathcal{B}$ and $\mathcal{P}$ being pushed to infinity.

1.2 Lagrangian and Hamiltonian formulations of gravity theory

1.2.1 Gravitational Lagrangian

Hilbert was the first to write down a Lagrangian for Einstein's General Theory of Relativity:

$$\mathfrak{L}_{\text{HE}} \stackrel{\text{def}}{=} \int_{\mathcal{M}} \sqrt{-g}\, R_{\mathcal{M}}, \tag{1.8}$$

thereby deducing the equations of motion for the gravitational field just before Einstein himself.

Since then, Einstein's theory has become like the trunk of a tree whence numerous theories branch off — some thicker, some thinner —, which in turn have other branches, and flowers sometimes. This manifold development of General Relativity is reflected in the many actions/Lagrangians of the branch theories, which may even be very different from one another, yet each always contains (1.8) as a particular case.

Among them, there are theories where the gravitational field is coupled to a scalar field, the dilaton, in a non-minimal way (the dilaton is multiplied by the curvature); such theories can often be, or actually are, derived from effective string theories, and represent their action at low energies.

In the present work, a Lagrangian of the latter kind will be considered:

$$\begin{aligned}\mathfrak{L} \stackrel{\text{def}}{=} \kappa \int_{\mathcal{M}} \sqrt{-\boldsymbol{g}}\, \eta(R_{\mathcal{M}} + \Lambda) + 2\kappa \int_{\mathcal{S}'}^{\mathcal{S}''} \sqrt{\boldsymbol{h}}\, \eta K \\ - 2\kappa \int_{\mathcal{B}} \sqrt{-\boldsymbol{\gamma}}\, \eta\Theta + 2\kappa \int_{\mathcal{P}'}^{\mathcal{P}''} \sqrt{\boldsymbol{\sigma}}\, \eta\alpha + \underline{\mathfrak{L}} + \mathfrak{L}^{\text{mat}};\end{aligned} \tag{1.9}$$

where η is the dilaton field, κ is a model- and dimension-dependent constant, Λ is twice the cosmological constant, $\underline{\mathfrak{L}} \stackrel{\text{def}}{=} \int_{\mathcal{B}} \sqrt{-\boldsymbol{g}}\, \underline{L}[\gamma_{ab}, \eta]$ is a boundary term which is a functional of the $\mathcal{B}$-induced metric (more precisely: geometry) and dilaton fields (so that it does not contribute to the equations of motion, but does contribute to the charges definition), $\mathfrak{L}^{\text{mat}}$ is the matter Lagrangian (minimally coupled to the gravitational field), and the other symbols have already been introduced. This Lagrangian is a particular case of the Brans-Dicke Lagrangian,

$$\begin{aligned}\mathfrak{L}_{\text{BD}} \stackrel{\text{def}}{=} \kappa \int_{\mathcal{M}} \sqrt{-\boldsymbol{g}}\, \eta \left(R_{\mathcal{M}} + \frac{\omega}{\eta^2} (\boldsymbol{\nabla}\eta)^2 + \Lambda \right) + 2\kappa \int_{\mathcal{S}'}^{\mathcal{S}''} \sqrt{\boldsymbol{h}}\, \eta K \\ - 2\kappa \int_{\mathcal{B}} \sqrt{-\boldsymbol{\gamma}}\, \eta\Theta + 2\kappa \int_{\mathcal{P}'}^{\mathcal{P}''} \sqrt{\boldsymbol{\sigma}}\, \eta\alpha + \underline{\mathfrak{L}} + \mathfrak{L}^{\text{mat}},\end{aligned} \tag{1.10}$$

and contains Hilbert-Einstein Lagrangian in turn. In what follows, we shall seldom be concerned with the matter term, but this will have no consequences upon the validity of the main reasoning lines and of final results.

The boundary terms make the Lagrangian (1.9) suited for a variational principle with fixation of the fields on the boundary; indeed the variation is:

$$\begin{aligned}\delta\mathfrak{L} = \int_{\mathcal{M}} (\boldsymbol{\Xi}^{\mu\nu} \delta g_{\mu\nu} + \boldsymbol{\Xi}^{\eta} \delta\eta) + \int_{\mathcal{S}'}^{\mathcal{S}''} (\boldsymbol{P}^{\mu\nu} \delta h_{\mu\nu} + \boldsymbol{P}^{\eta} \delta\eta) \\ + \int_{\mathcal{B}} (\boldsymbol{\Pi}^{\mu\nu} \delta\gamma_{\mu\nu} + \boldsymbol{\Pi}^{\eta} \delta\eta) + \int_{\mathcal{P}'}^{\mathcal{P}''} (\boldsymbol{\pi}^{\mu\nu} \delta\sigma_{\mu\nu} + \boldsymbol{\pi}^{\eta} \delta\eta) \\ + \int_{\mathcal{B}} (\underline{\boldsymbol{\Pi}}^{ab} \delta\gamma_{ab} + \underline{\boldsymbol{\Pi}}^{\eta} \delta\eta) + \int_{\mathcal{M}} \tfrac{1}{2} \sqrt{-\boldsymbol{g}}\, T^{\mu\nu} \delta g_{ab} \\ + \text{ [terms coming from the variation of the matter fields]},\end{aligned} \tag{1.11}$$

where the symbols have the following definitions:

$$\begin{aligned}\boldsymbol{\Xi}^{\mu\nu} \stackrel{\text{def}}{=} -\kappa\sqrt{-\boldsymbol{g}}\, [\eta G^{\mu\nu} - \tfrac{1}{2}\eta\Lambda g^{\mu\nu} + g^{\mu\nu} (\boldsymbol{\nabla}\eta)^2 - \boldsymbol{\nabla}^{\mu}\boldsymbol{\nabla}^{\nu}\eta] \\ = \sqrt{-\boldsymbol{g}}\, \Xi^{\mu\nu},\end{aligned} \tag{1.12a}$$

$$\boldsymbol{\Xi}^{\eta} \stackrel{\text{def}}{=} -\kappa\sqrt{-\boldsymbol{g}}\, (R_{\mathcal{M}} + \Lambda) = \sqrt{-\boldsymbol{g}}\, \Xi^{\eta}, \tag{1.12b}$$

$$\boldsymbol{P}^{\mu\nu} \stackrel{\text{def}}{=} -\kappa\sqrt{\boldsymbol{h}}\, [\eta(K^{\mu\nu} - K h^{\mu\nu}) + h^{\mu\nu} u^{\tau} \partial_{\tau}\eta] = \sqrt{\boldsymbol{h}}\, P^{\mu\nu}, \tag{1.12c}$$

$$\boldsymbol{P}^{\eta} \stackrel{\text{def}}{=} 2\kappa\sqrt{\boldsymbol{h}}\, K = \sqrt{\boldsymbol{h}}\, P^{\eta}, \tag{1.12d}$$

$$\boldsymbol{\Pi}^{\mu\nu} \stackrel{\text{def}}{=} -\kappa\sqrt{-\boldsymbol{\gamma}}\, [\eta(\Theta^{\mu\nu} - \Theta\gamma^{\mu\nu}) + \gamma^{\mu\nu} n^{\tau} \partial_{\tau}\eta] = \sqrt{-\boldsymbol{\gamma}}\, \Pi^{\mu\nu}, \tag{1.12e}$$

$$\mathbf{\Pi}^{\eta} \stackrel{\text{def}}{=} -2\kappa\sqrt{-\gamma}\,\Theta = \sqrt{-\gamma}\,\Pi^{\eta}, \tag{1.12f}$$

$$\boldsymbol{\pi}^{\mu\nu} \stackrel{\text{def}}{=} \kappa\sqrt{\boldsymbol{\sigma}}\,\alpha\eta\sigma^{\mu\nu} = \sqrt{\boldsymbol{\sigma}}\,\pi^{\mu\nu}, \tag{1.12g}$$

$$\boldsymbol{\pi}^{\eta} \stackrel{\text{def}}{=} 2\kappa\sqrt{\boldsymbol{\sigma}}\,\alpha = \sqrt{\boldsymbol{\sigma}}\,\pi^{\eta}, \tag{1.12h}$$

$$\underline{\mathbf{\Pi}}^{ab} \stackrel{\text{def}}{=} \frac{\delta \underline{L}}{\delta\gamma_{ab}} = \sqrt{-\gamma}\,\underline{\Pi}^{\mu\nu}, \tag{1.12i}$$

$$\underline{\mathbf{\Pi}}^{\eta} \stackrel{\text{def}}{=} \frac{\delta \underline{L}}{\delta\eta} = \sqrt{-\gamma}\,\underline{\Pi}^{\eta}, \tag{1.12j}$$

$$T^{\mu\nu} \stackrel{\text{def}}{=} \frac{2}{\sqrt{-\boldsymbol{g}}}\frac{\delta L^{\text{mat}}}{\delta g_{\mu\nu}}. \tag{1.12k}$$

Since we assumed that the metric and dilaton induced on the boundary were fixed, their variations on the boundary vanish and so do all terms but the first and the last in (1.11); in order for the first and last terms to vanish as well, we must set to zero the coefficients of $\delta g_{\mu\nu}$ and $\delta\eta$; thus we have the equations of motion:

$$\Xi^{\mu\nu} = -\tfrac{1}{2}T^{\mu\nu}, \tag{1.13a}$$

$$\Xi^{\eta} = 0. \tag{1.13b}$$

1.2.2 Gravitational Hamiltonian

A Hamiltonian formulation of gravity theory is appealing in view of its subsequent possible quantization, — and so in view of a quantum theory of gravity. Such a formulation is more or less well established today, and represents a way to a better understanding of gravity theory's principles; a consistent (divergenceless) quantization is not at hand yet, though.

The most important steps toward the construction of the gravitational Hamiltonian have been taken by Dirac [27], Arnowitt, Deser, Misner [3], DeWitt [26], and Teitelboim [39, 45], to say nothing of many others.

In the Hamiltonian formalism, the distinction is crucial between the coordinates that describe the system in phase space, and the time coordinate that traces the system evolution; the distinction between space and time for short. In the classical formulation of gravity theory, this distinction is almost completely suppressed instead — of course, since this is one of the theory's principles. Hence, in constructing a gravitational Hamiltonian, one must 'retrace one's steps' with respect to this principle, and restore the distinction between time and space.

The distinction is accomplished by foliating the spacetime manifold $\mathcal{M}$, where the metric field $g_{\mu\nu}$ lives, into spacelike hypersurfaces $\mathcal{S}$ (q.v. Sec. 1.1); this way the system's phase space is spanned by the spacelike metric components h_{ij} and the dilaton η which live on the leaves, and by their conjugate momenta $\boldsymbol{P}^{ij}$ and $\boldsymbol{P}^{\eta}$: these are the new dynamical variables. Thus the number of degrees of freedom decreases from $\frac{1}{2}(D^2+3D+4)$ to $\frac{1}{2}(D^2+D+2)$.

The system trajectory may be visualized as an evolution of the hypersurfaces — which carry the metric and dilaton fields — in a temporal direction, and the evolution take place between two fixed hypersurfaces $(\mathcal{S}_{t'}, h'_{ij}, \eta', \boldsymbol{P}^{ij\,\prime}, \boldsymbol{P}^{\boldsymbol{\eta}\,\prime})$ and $(\mathcal{S}_{t''}, h''_{ij}, \eta'', \boldsymbol{P}^{ij\,\prime\prime}, \boldsymbol{P}^{\boldsymbol{\eta}\,\prime\prime})$, in analogy with a classical system's path between two fixed points. The (classical) trajectory of the gravitational system must extremise the action, and this leads to the Hamiltonian equations of motion for the metric components, the dilaton and their momenta.

Mathematically, all this corresponds to a division of Einstein's equations into two groups: $D+1$ equations impose compatibility constraints on the initial data $(h'_{ij}, \eta', \boldsymbol{P}^{ij\,\prime}, \boldsymbol{P}^{\boldsymbol{\eta}\,\prime})$, while the remaining D^2 equations yield the effective dynamical evolution.

The Lagrangian $\mathfrak{L}$, which has a covariant, coordinate-independent form, can be re-expressed in a canonical form, in terms of the objects which emerge from the spacetime foliation (q.v. e.g. Kijowski [31]), by means of the Gauss-Codazzi relation

$$R_{\mathcal{M}} = R_{\mathcal{S}} - K^2 + K^{ij}K_{ij} - 2\boldsymbol{\nabla}_\mu(u^\mu K + u^\nu \boldsymbol{\nabla}_\nu u^\mu); \tag{1.14}$$

and it becomes

$$\begin{aligned} \mathfrak{L} = \mathfrak{S} \stackrel{\text{def}}{=} \int_{t'}^{t''} \Bigg[&\int_{\mathcal{S}} (\boldsymbol{P}^{\mu\nu}\dot{h}_{\mu\nu} + \boldsymbol{P}^{\boldsymbol{\eta}}\dot{\eta} - N\boldsymbol{H} - N^i\boldsymbol{H}_i) \\ &- \int_{\mathcal{P}} (\tilde{N}\boldsymbol{E} - \tilde{N}^A\boldsymbol{J}_A)\Bigg], \end{aligned} \tag{1.15}$$

where

$$\begin{aligned} \boldsymbol{H} &\stackrel{\text{def}}{=} 2\sqrt{\boldsymbol{h}}\, u_\mu u_\nu \Xi^{\mu\nu} \\ &= -2\boldsymbol{P}^{ij}K_{ij} + \boldsymbol{P}^{\boldsymbol{\eta}}u^\tau\partial_\tau\eta - \sqrt{\boldsymbol{h}}\,[\eta(R_{\mathcal{S}} + K^{ij}K_{ij} - K^2) \\ &\quad + 2Ku^\tau\partial_\tau\eta - \eta(u^\tau\partial_\tau\eta)^2 + \eta(\mathbf{D}\eta)^2 - 2\mathbf{D}^2\eta + \eta\Lambda] \end{aligned} \tag{1.16a}$$

is the Hamiltonian energy constraint, while

$$\boldsymbol{H}_i \stackrel{\text{def}}{=} -2\sqrt{\boldsymbol{h}}\, h_{i\mu}u_\nu\Xi^{\mu\nu} = -2\mathbf{D}_k\boldsymbol{P}^k_i + \boldsymbol{P}^{\boldsymbol{\eta}}\mathbf{D}_i\eta \tag{1.16b}$$

is the Hamiltonian momentum constraint; and

$$\boldsymbol{E} = 2\sqrt{\boldsymbol{\sigma}}\,\tilde{u}_a\tilde{u}_b\Pi^{ab} = 2\sqrt{\boldsymbol{\sigma}}\,(-n^\mu\boldsymbol{\nabla}_\mu\eta + \eta k) - \underline{\boldsymbol{E}}, \tag{1.17a}$$

$$\boldsymbol{J}_A = 2\sqrt{\boldsymbol{\sigma}}\,\sigma_{Aa}\tilde{u}_b\Pi^{ab} = 2\sqrt{\boldsymbol{\sigma}}\,\sigma_{Ai}n_jP^{ij} - \underline{\boldsymbol{J}}_A, \tag{1.17b}$$

with

$$\underline{\boldsymbol{E}} = 2\sqrt{\boldsymbol{\sigma}}\,\tilde{u}_a\tilde{u}_b\underline{\Pi}^{ab}, \tag{1.17c}$$

$$\underline{\boldsymbol{J}}_A = 2\sqrt{\boldsymbol{\sigma}}\,\sigma_{Aa}\tilde{u}_b\underline{\Pi}^{ab}, \tag{1.17d}$$

are the energy and momentum boundary terms.

We shall some important remarks upon the boundary integral in Eq. (1.15) later; now we just want to note that the Hamiltonian energy and momentum constraints $\boldsymbol{H}$ and $\boldsymbol{H}_i$ depend upon the phase-space variables $(h_{ij}, \eta, \boldsymbol{P}^{ij}, \boldsymbol{P}^{\boldsymbol{\eta}})$ and the lapse and shift; the quantities $\boldsymbol{E}$ and $\boldsymbol{J}_i$, instead, do depend solely upon the canonical variables *only if* $\underset{\sim}{\boldsymbol{E}}$ and $\underset{\sim}{\boldsymbol{J}}_i$ do (we shall see later that these quantities are the system energy and momentum).

By means of a Legendre transformation

$$\int_{t'}^{t''} \mathfrak{H} = \int_{t'}^{t''} \int_{\mathcal{S}} (\boldsymbol{P}^{\mu\nu} \dot{h}_{\mu\nu} + \boldsymbol{P}^{\boldsymbol{\eta}} \dot{\eta}) - \mathfrak{L} \tag{1.18}$$

one finally obtains the gravitational Hamiltonian

$$\mathfrak{H} \stackrel{\text{def}}{=} \int_{\mathcal{S}} (N\boldsymbol{H} + N^i \boldsymbol{H}_i) + \int_{\mathcal{P}} (\tilde{N}\boldsymbol{E} - \tilde{N}^A \boldsymbol{J}_A), \tag{1.19}$$

and the canonical equations of motion are

$$\dot{h}_{ij} = \frac{\delta \mathfrak{H}}{\delta \boldsymbol{P}^{ij}}, \tag{1.20a}$$

$$\dot{\eta} = \frac{\delta \mathfrak{H}}{\delta \boldsymbol{P}^{\boldsymbol{\eta}}}, \tag{1.20b}$$

$$\dot{\boldsymbol{P}}^{ij} = -\frac{\delta \mathfrak{H}}{\delta h_{ij}}, \tag{1.20c}$$

$$\dot{\boldsymbol{P}}^{\boldsymbol{\eta}} = -\frac{\delta \mathfrak{H}}{\delta \eta}. \tag{1.20d}$$

This kind of Hamiltonian approach, which adopts $\{h_{ij}, \eta, \boldsymbol{P}^{ij}, \boldsymbol{P}^{\boldsymbol{\eta}}, N, N^i\}$ as canonical variables, is mainly due to Arnowitt, Deser and Misner [3]. There are also other approaches, which use a different set of variables and present other advantages, like e.g. Ashtekar's approach [4] that uses a spinor structure on the spacelike leaves. In the present work only Arnowitt, Deser and Misner's method will be considered.

1.2.3 The lapse and shift

The so called 'lapse' N and 'shift' N^i in Eqs. (1.15) and (1.19) play the role of Lagrange multipliers; indeed, the vanishing of the coefficients of their variations leads to the equations $\boldsymbol{H} = 0$ and $\boldsymbol{H}_i = 0$, which are just the constraint equations for the initial data. A very important feature of the gravitational Hamiltonian is that the lapse and shift are not determined by the equations of motion, but rather they are to be specified *ab initio* in order to integrate the equations (a similar situation arises, with the scalar potential, in the Hamiltonian for the electromagnetic field; q.v. Misner, Thorne e Wheeler [37, §21.8]). This characteristic appears as a residual of the Lagrangian full gauge symmetry, and is related to the fact that there is not only one possible foliation of $(\mathcal{M}, g_{\mu\nu})$

between the hypersurfaces $(\mathcal{S}_{t'}, h'_{ij})$ and $(\mathcal{S}_{t''}, h''_{ij})$, but rather an infinite family of foliations; only after one of them is given, can the canonical variables' evolution be studied. When one specifies the lapse and shift, one is in fact specifying a foliation; more precisely, one is specifying the unique future-pointing, timelike unit vector field u^μ which is normal to every leaf of that foliation. The vector components are given by:

$$u^0 = \frac{1}{N}, \tag{1.21a}$$

$$u^i = -\frac{N^i}{N}. \tag{1.21b}$$

The lapse $N(t, x^i)$ determines the lapse of proper time, which amounts to $N(t, x^i)\,\delta t$, from the point (x^i) on the hypersurface $\mathcal{S}_t$ to the point $(x^i + \delta x^i)$ on $\mathcal{S}_{t+\delta t}$; the shift $N^i(t, x^i)$ determines the tangential shift of the same point, which amounts to $\delta x^i \equiv N^i(t, x^i)\,\delta t$.

A quite interesting feature of this function and vector field is that they can be used to study how the system evolves along a series of hypersurfaces given the action of a one-parameter transformation group $\mathbf{T}_t$: the hypersurface $\mathcal{S}_{t+\delta t}$ is given by $\mathcal{S}_{t+\delta t} = \mathbf{T}_{\delta t}\mathcal{S}_t = \mathbf{L}_{(\delta t\,\xi)}\mathcal{S}_t$, where ξ is the group generator. The lapse and shift corresponding to the foliation thus generated can be expressed as:

$$N \stackrel{\text{def}}{=} -\xi^\mu u_\mu = \frac{1}{\sqrt{-g^{00}}}\xi^0, \tag{1.22a}$$

$$N^i \stackrel{\text{def}}{=} \xi^\mu h^i{}_\mu = \xi^i - \frac{g^{0i}}{g^{00}}\xi^0. \tag{1.22b}$$

1.2.4 The boundary integral

The above-given expression of the boundary integral in Eq. (1.15),

$$\int_{\mathcal{P}} (\tilde{N}\boldsymbol{E} - \tilde{N}^A \boldsymbol{J}_A), \tag{1.23a}$$

with

$$\boldsymbol{E} = 2\sqrt{\boldsymbol{\sigma}}\,\tilde{u}_a\tilde{u}_b\Pi^{ab} = 2\sqrt{\boldsymbol{\sigma}}\,(-n^\mu\boldsymbol{\nabla}_\mu\eta + \eta k) - \underline{\boldsymbol{E}}, \tag{1.23b}$$

$$\boldsymbol{J}_A = 2\sqrt{\boldsymbol{\sigma}}\,\sigma_{Aa}\tilde{u}_b\Pi^{ab} = 2\sqrt{\boldsymbol{\sigma}}\,\gamma_{Ai}n_jP^{ij} - \underline{\boldsymbol{J}}_A, \tag{1.23c}$$

$$\underline{\boldsymbol{E}} = 2\sqrt{\boldsymbol{\sigma}}\,\tilde{u}_a\tilde{u}_b\underline{\Pi}^{ab}, \tag{1.23d}$$

$$\underline{\boldsymbol{J}}_A = 2\sqrt{\boldsymbol{\sigma}}\,\sigma_{Aa}\tilde{u}_b\underline{\Pi}^{ab}, \tag{1.23e}$$

is only valid when the following equation holds:

$$\tilde{N}\tilde{u}^\mu + \tilde{N}^\mu = Nu^\mu + N^\mu \tag{1.24}$$

(q.v. e.g. Booth and Mann [11]). The meaning of condition (1.24) becomes immediately clear upon noticing that the vector field $Nu^\mu + N^\mu$ generates the evolution of the hypersurfaces $\mathcal{S}$, and the vector field $\tilde{N}\tilde{u}^\mu + \tilde{N}^\mu$ is always tangent to $\mathcal{B}$: Eq. (1.24) is equivalent to requiring that the initial surface $\mathcal{P}' = \partial\mathcal{S}'$ should evolve tangentially to the boundary $\mathcal{B}$, i.e. neither 'crashing into' nor 'coming out of' $\mathcal{B}$ (this is a legitimate request when $\mathcal{B}$ is the actual boundary of the spacetime, but it is not when $\mathcal{B}$ is just a temporary, 'fictitious' boundary to be eventually pushed to infinity).

In addition, if the hypersurfaces $\mathcal{S}$ are always orthogonal to $\mathcal{B}$, one also has, in the adapted coordinate system:

$$\tilde{u}^\mu = u^\mu, \tag{1.25a}$$

$$\tilde{N} = N, \tag{1.25b}$$

$$\tilde{N}^A = N^A, \tag{1.25c}$$

$$N^r = 0, \tag{1.25d}$$

so that the boundary integral (1.23) can be written as:

$$\int_{\mathcal{P}} (N\boldsymbol{E} - N^A \boldsymbol{J}_A) \tag{1.26a}$$

with

$$\boldsymbol{E} = 2\sqrt{\boldsymbol{\sigma}}\, u_a u_b \Pi^{ab} = 2\sqrt{\boldsymbol{\sigma}}\, (-n^\mu \boldsymbol{\nabla}_\mu \eta + \eta k) - \underline{\boldsymbol{E}}, \tag{1.26b}$$

$$\boldsymbol{J}_A = 2\sqrt{\boldsymbol{\sigma}}\, \sigma_{Aa} u_b \Pi^{ab} = 2\sqrt{\boldsymbol{\sigma}}\, h_{Ai} n_j P^{ij} - \underline{\boldsymbol{J}}_A, \tag{1.26c}$$

$$\underline{\boldsymbol{E}} = 2\sqrt{\boldsymbol{\sigma}}\, u_a u_b \underline{\Pi}^{ab}, \tag{1.26d}$$

$$\underline{\boldsymbol{J}}_A = 2\sqrt{\boldsymbol{\sigma}}\, \sigma_{Aa} u_b \underline{\Pi}^{ab}. \tag{1.26e}$$

(q.v. e.g. Brown and York [17], Creighton and Mann [24]).

If one gives up condition (1.24), then the expression of the boundary integral, *for a non-dilatonic theory*, becomes (q.v. Hawking and Hunter [30]):

$$\int_{\mathcal{P}} (N\boldsymbol{E} - N^i \boldsymbol{J}_i) \tag{1.27a}$$

with

$$\boldsymbol{E} \stackrel{\text{def}}{=} \sqrt{\boldsymbol{\sigma}} \left(2k - 2\frac{\alpha}{\cosh\alpha} \boldsymbol{\nabla}_\mu \tilde{u}^\mu \right) - \underline{\boldsymbol{E}}, \tag{1.27b}$$

$$\boldsymbol{J}_i \stackrel{\text{def}}{=} 2\sqrt{\boldsymbol{\sigma}}\, \tilde{n}^j P_{ji} - \underline{\boldsymbol{J}}_i. \tag{1.27c}$$

Who writes has not found in the literature, nor has calculated personally, the generalisation of Eqs. (1.27) for a dilatonic theory.[1]

[1] *Note added in translation:* We eventually calculated such a generalisation, q.v. *Hamiltonians for a general dilaton gravity theory on a spacetime with a non-orthogonal, timelike or spacelike outer boundary*, to appear in *Class. Quantum Gravity.*

1.3 Anti-de Sitter space and black holes

1.3.1 Anti-de Sitter space

Definition

The $(D{+}1)$-dimensional anti-de Sitter space is a pseudo-Riemannian manifold with negative constant curvature $-\lambda^2$, and can be easily constructed starting from the hyperboloid

$$(y^0)^2 + (y^{D+1})^2 - \sum_{i=1}^{D} (y^i)^2 = \lambda^{-1} \tag{1.28}$$

in flat space $(\mathbb{R}^{D+2}, \hat{\eta})$, where $\hat{\eta}$ is the metric

$$\hat{\eta} \stackrel{\text{def}}{=} -(dy^0)^2 - (dy^{D+1})^2 + \sum_{i=1}^{D} (dy^i)^2. \tag{1.29}$$

By construction, the hyperboloid shares the same group of isometries of the embedding space (except for translations), $SO(2, D)$.

The hyperboloid (1.28) can be described parametrically by

$$y^0 = \lambda^{-1} \cosh\rho \, \cos\tau \tag{1.30a}$$
$$y^i = \lambda^{-1} \sinh\rho \, \Omega^i \qquad (i = 1, \ldots, D) \tag{1.30b}$$
$$y^{D+1} = \lambda^{-1} \cosh\rho \, \sin\tau \tag{1.30c}$$

with

$$\rho \geq 0, \tag{1.31a}$$
$$0 \leq \tau < 2\pi, \tag{1.31b}$$
$$\Omega^i \text{ coordinates on } S^D, \tag{1.31c}$$

so that the intrinsic metric is

$$ds^2 = \lambda^{-2}(-\cosh^2\rho \, d\tau^2 + d\rho^2 + \sinh^2\rho \, d\Omega^2), \tag{1.32}$$

where $d\Omega^2$ is the metric on S^D.

The hyperboloid's topology is $S^1 \times \mathbb{R}^D$, so that timelike curves are present; in order to have a causal spacetime, one passes to the universal covering, which has topology $\mathbb{R}^1 \times \mathbb{R}^D$ and the same metric (1.32) but with new coordinate intervals:

$$\rho \geq 0, \tag{1.33a}$$
$$-\infty < \tau < +\infty, \tag{1.33b}$$
$$\Omega^i \text{ coordinates on } S^D. \tag{1.33c}$$

By definition, $(D+1)$-dimensional anti-de Sitter space is this covering.

One can always find a coordinate chart (t, Ω^i, r) such that the metric takes the following form:

$$ds^2 = -(\lambda^2 r^2 + 1)\, dt^2 + (\lambda^2 r^2 + 1)^{-1}\, dr^2 + r^2\, d\Omega^2; \tag{1.34}$$

such chart does not cover the whole manifold, but it will be very useful in the asymptotic analysis of black-hole solutions.

Causal properties

The causal structure of anti-de Sitter space presents interesting features. By means of the coordinate change

$$\theta = \arctan(\sinh\rho) \qquad \left(0 \le \theta < \frac{\pi}{2}\right), \tag{1.35}$$

the metric (1.32) becomes

$$ds^2 = \frac{1}{\lambda^2 \cos^2\theta}(-d\tau^2 + d\theta^2 + \sinh^2\theta\, d\Omega^2); \tag{1.36}$$

thus we see that anti-de Sitter space is conformal to the interior of the cylinder $\mathbb{R}\times S^{D-1}$, and its boundary (the cylinder) is timelike (Fig. 1.2) — as opposed e.g. to the boundary of (conformally compactified) Minkowski space, which is null.

This boundary characterizes all *asymptotically anti-de Sitter* solutions of the theory, e.g. black-hole solutions, and possesses two important (causal) features: first, it is conformal to D-dimensional compactified Minkowski space (short of the addition of two compactifying points at infinity $\tau = -\infty$ and $\tau = +\infty$); second, an observer can see a light signal going to infinity and coming back in a *finite* lapse of his proper time.

The first feature allows a D-dimensional conformal field theory to live on the boundary; the second feature holds some important consequences for black-hole solutions, for it allows one to have a thermal bath by a finite amount of energy, and hence stable solutions (in thermal equilibrium).

1.3.2 Black holes in anti-de Sitter space

Fundamental properties

Einstein's equation in four dimensions admits the well-known Schwarzschild's black-hole metric as a solution:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2, \tag{1.37}$$

where M is the mass of the black hole. The characteristics of this solution which are of interest to us are the following:

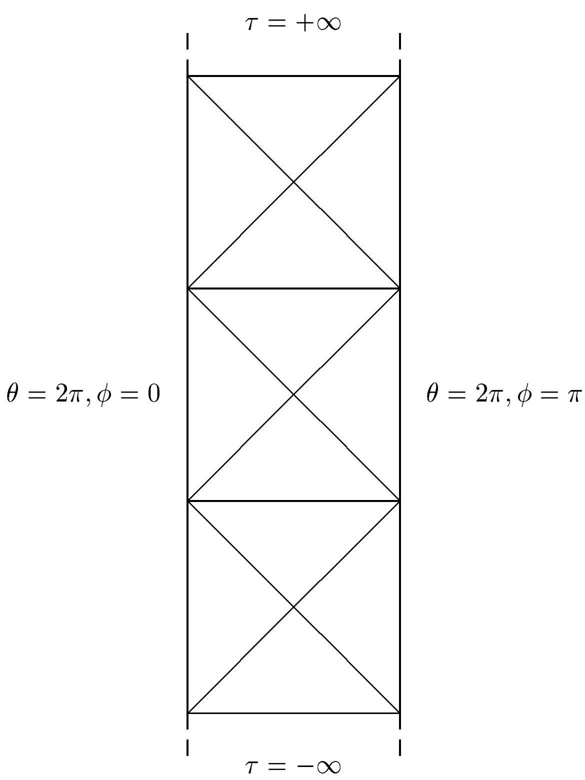

Figure 1.2: A section of the Penrose diagram for anti-de Sitter space.

1. the presence of an asymptotically flat region[2], i.e. of a boundary like that of Minkowski space;
2. spherical symmetry and stationarity[3], i.e. the invariance under the group $SO(3)$ (or $SO(D)$ in $(D+1)$ dimensions) and the existence of a timelike Killing vector field;
3. the presence of a spacelike polynomial singularity, i.e. a spacelike hypersurface where the curvature diverges, which makes the manifold geodetically incomplete;
4. the presence of an event horizon, i.e. a null hypersurface from whose interior no physical signal is (classically) allowed to escape[4].

Considering a more general gravity theory with a cosmological constant and a dilaton, we should like to find a solution with similar characteristics, but that e.g. should have an asymptotic region with non-zero constant curvature — i.e. that should be asymptotically anti-de Sitter. Besides, we should like to study it in two or three dimensions, thus making the computational machinery easier (q.v. Lemos [34]).

The three-dimensional case

Dilatonic theory In three dimensions, a dilatonic theory with Lagrangian

$$\frac{1}{2\pi}\int_{\mathcal{M}} d^3x\,\sqrt{-\boldsymbol{g}}\,\eta(R_{\mathcal{M}}+6\lambda^2) \tag{1.38}$$

(where boundary terms have been omitted) possesses a class of solutions similar to (1.37) (q.v. Cadoni [18]):

$$ds^2 = -\left(\lambda^2 r^2 - \frac{\alpha^2}{r}\right)dt^2 + \left(\lambda^2 r^2 - \frac{\alpha^2}{r}\right)^{-1} dr^2 + r^2 d\phi^2, \tag{1.39a}$$

$$\eta = \lambda r; \tag{1.39b}$$

it describes a spacetime which is asymptotically anti-de Sitter (in the sense that its curvature is asymptotically constant and negative), spherically symmetric, stationary, and with a polynomial singularity and an event horizon: hence we can interpret it as a black-hole solution in anti-de Sitter space; the black-hole mass is $M = 2\lambda\alpha^2$, and the ground state is

$$ds^2 = -\lambda^2 r^2 dt^2 + (\lambda^2 r^2)^{-1} dr^2 + r^2 d\phi^2, \tag{1.40a}$$

$$\eta = \lambda r. \tag{1.40b}$$

[2] We shall not consider the maximal analytical extension of the solution (1.37), given by the Kruskal-Szekeres metric, which has two distinct asymptotic regions; only one asymptotic region is of interest to us.

[3] In the absence of matter the second property follows from the first by Birkhoff's theorem.

[4] We shall not adopt the definition of an event horizon as a separation hypersurface between two distinct asymptotic regions; see Note 2.

Non-dilatonic theory However, we face some difficulties as soon as we consider a non-dilatonic theory. In three dimensions, indeed, an asymptotically anti-de Sitter black-hole solution similar to (1.37) exists for the non-dilatonic action in four dimensions

$$\frac{1}{16\pi}\int_{\mathcal{M}} d^4x\,\sqrt{-\boldsymbol{g}}\,(R_{\mathcal{M}}+6\lambda^2), \tag{1.41}$$

namely

$$ds^2=-\left(\lambda^2r^2+1-\frac{2M}{r}\right)dt^2+\left(\lambda^2r^2+1-\frac{2M}{r}\right)^{-1}dr^2+r^2d\Omega^2, \tag{1.42}$$

but such a solution exists only in a number of dimensions greater than three.

Bañados, Teitelboim, and Zanelli [8] were the first to construct a three-dimensional asymptotically anti-de Sitter black-hole solution, by simply altering the *global* topology of anti-de Sitter space (q.v. also Bañados, Henneaux, Teitelboim, Zanelli [7], Bañados, Gomberoff, Martínez [6]). The alteration is realized by taking the quotient space of the action of a particular transformation group which acts over anti-de Sitter space, i.e. by identifying some points; to be more precise, in the coordinate system where the metric is given by:

$$ds^2=-(\lambda^2r^2+1)dt^2+(\lambda^2r^2+1)^{-1}dr^2+r^2d\phi^2, \tag{1.43}$$

the points (t,r,ϕ) and $(t-2\pi A,r,\phi+2\pi\alpha)$ are identified. This amounts to removing an angular slice equal to $2\pi(1-\alpha)$ and inserting a time jump equal to $2\pi A$.

The space thus obtained is locally exactly alike to the original one: it has negative constant (not only *asymptotically* constant) curvature, isometry group $\mathbb{R}\times SO(2)$, and it is a solution for the equations of motion coming from the following Lagrangian:

$$\frac{1}{2\pi}\int_{\mathcal{M}} d^3x\,\sqrt{-\boldsymbol{g}}\,(R_{\mathcal{M}}+2\lambda^2). \tag{1.44}$$

The point identification gives rise to a region which hosts closed timelike curves, so that one has to cut this region away, thus rendering the spacetime geodetically incomplete, and the (spacelike) cut-surface can be viewed as a singularity; it is just a 'causal' singularity, and not a polynomial one, because the curvature tensors do not diverge near it. When $|A|<\alpha\lambda$ the singularity is hidden by a null hypersurface from whose interior no signal may escape, so that it can be viewed as an event horizon.

Summing up, we have a solution with the following characteristics:

1. it is asymptotically anti-de Sitter, since it is *locally* anti-de Sitter[5];

[5]Moreover, its maximal analytical extension has *only one* asymptotic region, as opposed to the Schwarzschild solution's extension, which has two (q.v. Note 2).

2. it is circularly symmetric (i.e. $SO(2)$-invariant) and stationary;
3. it has a spacelike causal singularity;
4. it has an event horizon.

It can naturally be considered a black-hole solution; it is called a 'topological black hole', for the way it is obtained. The parameter $M = \alpha$ is the black-hole mass and $J = \alpha A$ is the black-hole angular momentum.

In order to have a coordinate system without jumps, one makes the following coordinate transformation:

$$t \longmapsto \alpha t - A\phi \tag{1.45a}$$

$$r \longmapsto r(\alpha^2 - A^2\lambda^2)^{-\frac{1}{2}} \tag{1.45b}$$

$$\phi \longmapsto \alpha\phi - A\lambda^2 t \tag{1.45c}$$

and the new expression for the metric (1.43) is:

$$\begin{aligned} ds^2 = &-(\lambda^2 r^2 + \alpha^2)dt^2 + (\lambda^2 r^2 + \alpha^2 - A^2\lambda^2)^{-1}dr^2 \\ &+ 2\alpha A dt\, d\phi + (r^2 - A^2)d\phi^2. \end{aligned} \tag{1.46}$$

In this coordinate system, the isometries are generated by the Killing vector fields $\partial/\partial t$ and $\partial/\partial\phi$.

Both the dilatonic and the non-dilatonic black-hole solutions, Eqs. (1.39) and (1.46) respectively, share the same Penrose diagram for the ground state ($M = 0$) and for a black hole with positive mass ($M > 0$) (Figs. 1.3 and 1.4).

The bidimensional case

A gravity theory in two dimensions must necessarily be a dilaton one, because the curvature is a topological invariant in two dimensions: an action like the Hilbert-Einstein action (with or without a cosmological constant) would not have any dynamics without the introduction of one more degree of freedom, represented by the dilaton field (indeed, General Relativity in two dimensions has -1 effective degrees of freedom).

Hence one adopts Jackiw and Teitelboim's Lagrangian,

$$\frac{1}{2}\int_{\mathcal{M}} d^2x\,\sqrt{-\boldsymbol{g}}\,(R_{\mathcal{M}} + 2\lambda^2), \tag{1.47}$$

whose equations of motion possess black-hole solutions of a 'topological' kind; they have the form:[6]

$$ds^2 = -(\lambda^2 r^2 + \alpha^2)dt^2 + (\lambda^2 r^2 + \alpha^2)^{-1}dr^2, \tag{1.48a}$$

$$\eta = \eta_0 \lambda r. \tag{1.48b}$$

[6] The arbitrary constant η_0 shall be often set to unity in the following sections.

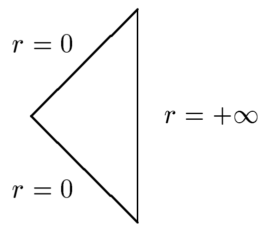

Figure 1.3: A section of the Penrose diagram for the ground state ($M = 0$) solution in three-dimensional anti-de Sitter space.

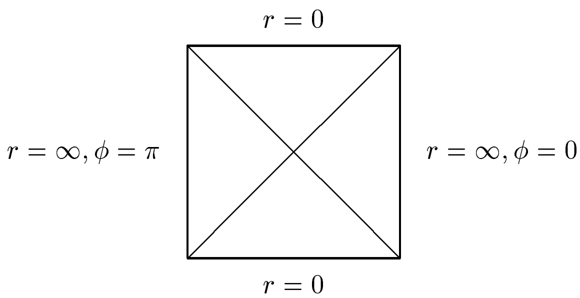

Figure 1.4: A section of the Penrose diagram for the positive-mass black-hole solution ($M > 0$) in three-dimensional anti-de Sitter space.

The metrics (1.48a), contrary to what one could expect because of the presence of the parameter α, are just different parameterizations of two-dimensional anti-de Sitter space, covering different regions; it is just the presence of the dilaton which makes them geometrically and physically different. The dilaton must be positive, since it plays the role of a coupling constant; this implies that we must cut the hypersurface where the dilaton vanishes and remove the region where it is negative, thus the manifold becomes geodetically incomplete and a causal singularity appears. The cut-surface can be of different causal kinds, depending on the parameter α: it is timelike when $\alpha^2 < 0$, null when $\alpha^2 = 0$, and spacelike when $\alpha^2 > 0$; in the latter case an event horizon is also present. Thus, we have three families of solutions with different *global* topologies, which we call AdS_2^- ($\alpha^2 < 0$), AdS_2^0 ($\alpha^2 = 0$), and AdS_2^+ ($\alpha^2 > 0$). The solution with $\alpha^2 = 0$ can be considered as the ground state, and that with $\alpha^2 > 0$ as a black-hole solution with positive mass $M = \frac{1}{2}\eta_0\lambda\alpha^2$: its characteristics are similar to Bañados, Teitelboim and Zanelli's solution:

1. it is asymptotically and locally AdS_2^0;

2. it is stationary[7];

3. it has a spacelike causal singularity;

4. it has an event horizon.

One should note that we have not said 'asymptotically anti-de Sitter', since full anti-de Sitter space is not considered among the solutions, and its role as a ground state is played by AdS_2^0. Moreover, one should also note that the two-dimensional topological black-hole has *two* distinct asymptotic regions, as opposed to the three-dimensional black-hole, which has one (q.v. preceding notes); this is simply due to the fact that the sphere S^1 is not connected, while S^2 is. Apart from that, the Penrose diagrams for the ground state and the black-hole solutions are similar to those for the three-dimensional solutions (Figs. 1.5 and 1.6).

Dimensional reductions

Under certain symmetry conditions, it is possible to reduce the Lagrangian of a gravitational model to the Lagrangian of another gravitational model living in a less number of dimensions; one immediately understands that this can be very useful for studying high-dimensional models, since solutions for the lower-dimensional Lagrangian shall also be solutions for the higher-dimensional one.

The bidimensional Jackiw-Teitelboim Lagrangian and black-hole solutions are just an example of this dimensional reduction: they can indeed be obtained from a four-dimensional Lagrangian with dilaton and electromagnetic fields (related to a Brans-Dicke Lagrangian, q.v. Cadoni and Mignemi [21]), or from a

[7] One might also say that it is spherically symmetric, but that would be a triviality, since $SO(1)$ contains only the identity element.

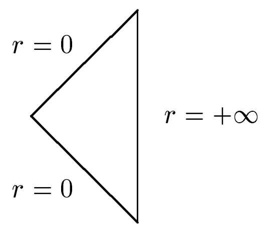

Figure 1.5: Penrose diagram for the ground state ($M = 0$) solution in bidimensional anti-de Sitter space.

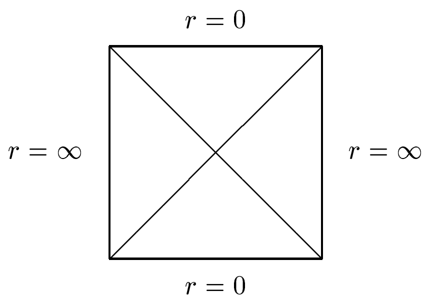

Figure 1.6: Penrose diagram for the black-hole solution with positive mass ($M > 0$) in bidimensional anti-de Sitter space.

three-dimensional Lagrangian whose solutions are just the Banãdos-Teitelboim-Zanelli black holes (q.v. Achúcarro and Ortiz [1]).

So, suppose that a three-dimensional metric, obtained from a variational principle with Lagrangian

$$\mathfrak{L} = \int_{\mathcal{M}} d^3x \sqrt{-\boldsymbol{g}}\,(R_{\mathcal{M}} + \Lambda), \tag{1.49}$$

has components which do not depend on a coordinate ϕ:

$$ds^2 = g_{\mu\nu}\,dx^\mu\,dx^\nu = f_{\alpha\beta}(x^\alpha)\,dx^\alpha\,dx^\beta + \Phi^2(x^\alpha)\,d\phi^2; \tag{1.50}$$

then, it is easy to verify that the Lagrangian (1.49) can be reduced to the bidimensional Jackiw-Teitelboim one:

$$\mathfrak{L} = \int_{\mathcal{M}} d^2x \sqrt{-\boldsymbol{f}}\,\Phi(R_{\mathcal{M}} + \Lambda). \tag{1.51}$$

As a consequence, every three-dimensional solution (1.50) corresponds to a bidimensional solution

$$ds^2 = f_{\alpha\beta}(x^\alpha)\,dx^\alpha\,dx^\beta, \tag{1.52a}$$

$$\eta = \Phi(x^\alpha). \tag{1.52b}$$

Just this correspondence holds between solutions (1.46) and (1.48a) (when $A = 0$); from this point of view, the dilaton is nothing but the component $g_{\phi\phi}$ of the three-dimensional metric, which is singular (coordinate singularity) where that component vanishes. This can be seen as another reason for removing the regions wherein the dilaton is not positive.

Thermodynamics and entropy

As a very short sketch of black-hole thermodynamics, consider a black hole described by the following parameters: mass M, angular momentum J, electric charge Q; then, the following relation holds among their variations:

$$\delta M = \frac{k}{2\pi}\,\delta\left(\frac{A}{4}\right) + \Omega\,\delta J + \Phi\,\delta Q, \tag{1.53}$$

where A is the area of the event horizon, k is the surface gravity, Ω is the angular velocity, and Φ is the electric potential; moreover, one also has, for a static solution, that

$$\frac{k}{2\pi} = \text{ constant, throughout the event horizon,} \tag{1.54}$$

and that (Hawking's theorem)

$$\delta\left(\frac{A}{4}\right) \geq 0. \tag{1.55}$$

The similarity between these three relations and the three thermodynamics laws is evident: a system in an equilibrium state has a constant quantity, namely the temperature T; its entropy S can only increase; for a system transformation the relation

$$\delta E = T\,\delta S + \Omega_i\,\delta W^i, \tag{1.56}$$

holds, where $\Omega_i\,\delta W^i$ are work terms.

The identification between the quantities

$$M = E, \tag{1.57a}$$

$$T = \frac{k}{2\pi}, \tag{1.57b}$$

$$S = \frac{A}{4}, \tag{1.57c}$$

which is only a formal one at a classical level, was shown to be a physical one as well at a semiclassical level, thanks to Hawking's investigations on black-hole evaporation.

Classical system's temperature and entropy can be given a statistical meaning in terms of microstates, and one would like to give a similar statistical meaning to black hole's as well; it appears that this will be achieved when a quantum theory of gravity will be at hand. Anti-de Sitter space proves to be very interesting in this context: we shall see that three-dimensional gravity on anti-de Sitter space and its black-hole solutions can be associated with a two-dimensional conformal dual theory whereupon statistical analysis can be done. A very important result is that the statistical entropy, calculated through the dual theory, exactly equals the thermodynamic one, which comes from the gravity theory.

Chapter 2

Asymptotic symmetries in the Hamiltonian and quasilocal formalisms

2.1 Asymptotic conditions and asymptotic symmetries

When a (gravitational) system's surface at infinity enjoys some symmetries, we speak about asymptotic symmetries. To have a clearer idea of this, we ought to state what we mean by 'surface at infinity' of a gravitational system and by 'symmetries of a surface at infinity'.

2.1.1 Surface at infinity and asymptotic conditions

In physics, the distinction between the bulk and the surface of a system is often very important. In gravity theory, the system usually consists of a differentiable manifold $\mathcal{M}$ and of all the geometro-physical objects that are defined therein; in this case the distinction between 'bulk' and 'surface' is very alike to (if we do not want to say "coincident with") the topological distinction between 'interior' and 'boundary', represented by the symbols '$\overset{\circ}{\mathcal{M}}$' and '$\partial\mathcal{M}$'. Nonetheless we prefer to use the terms 'bulk' and 'surface' rather than 'interior' and 'boundary', because we shall speak about a 'surface' even when the corresponding topological concept 'boundary' is ill- or non- or just intuitively-defined.

This comparison with mathematical terminology is a cue for clarifying the concept of 'surface at infinity'.

Compact manifold

Let $\mathcal{M}$ be a $(D+1)$-dimensional compact manifold with boundary, with metric g; the notion of surface or boundary $\partial\mathcal{M}$ is (topologically) well-defined[1]: we can define it as a (regular) embedding F of a D-dimensional manifold $\mathcal{B}$ into $\mathcal{M}$,

$$F : \mathcal{B} \longrightarrow \mathcal{M}, \quad F(\mathcal{B}) = \partial\mathcal{M}. \tag{2.1}$$

locally, it is always possible to choose a coordinate system $(x^0, \ldots, x^D) \equiv (x^a)$ on $\mathcal{B}$ and $(x^0, \ldots, x^D, r) \equiv (x^a, r) \equiv (x^\mu)$ on $\mathcal{M}$ such that the embedding in coordinates reads:

$$F : (x^0, \ldots, x^D) \longmapsto (x^0, \ldots, x^D, R), \quad R = \text{const}. \tag{2.2}$$

The embedding induces all various tangent maps, push-forwards and pull-backs between the various tangent bundles and tensor-field spaces on $\mathcal{B} \equiv \partial\mathcal{M}$ and $\mathcal{M}$, e.g.

$$TF : T\mathcal{M} \to T\mathcal{B} \tag{2.3a}$$

between the vector tangent bundles,

$$F_* : \mathfrak{X}(\mathcal{B}) \to \mathfrak{X}(\partial\mathcal{M}) \tag{2.3b}$$

between the vector-field spaces, or

$$F^* : \Lambda(\mathcal{M}) \to \Lambda(\mathcal{B}) \tag{2.3c}$$

between the form spaces (q.v. e.g. Choquet-Bruhat, De Witt-Morette, and Dillard-Bleick [23] or Bishop and Goldberg [10] or Curtis and Miller [25] or Schutz [40] or Sternberg [41]). Thus the metric γ_{ab} and a scalar field $\eta|_\mathcal{B}$ induced on $\mathcal{B}$ are defined as

$$\gamma_{ab} = F^*(g_{\mu\nu}) = g_{ab}|_{r=R}, \tag{2.4a}$$
$$\eta|_\mathcal{B} = F^*(\eta) = \eta|_{r=R}, \tag{2.4b}$$

and every tangent vector $\vec{v}$ on $\mathcal{B}$ can be seen as a tangent vector v on $\mathcal{M}$:

$$v^\mu = TF(\vec{v}^a) = \begin{pmatrix} \vec{v}^a \\ 0 \end{pmatrix}. \tag{2.5}$$

Non-compact manifold

Now let $\mathcal{M}$ be a *non*-compact, infinitely extended manifold instead; first of all, from a strict topological point of view, the boundary $\partial\mathcal{M}$ *does not exist at all* (it is the empty set), and we cannot speak about embeddings onto $\partial\mathcal{M}$. Rather, we use *limit* and series expansion as 'tools' here.

[1] We suppose we are dealing with non-pathological manifolds, of course.

In this case indeed, by 'surface' we mean an asymptotic region, which can be characterized e.g. by the coordinate $r \to \infty$. More subtle is defining the geometro-physical objects 'induced' on this 'surface at infinity'. At first sight, for example, it would seem reasonable to define the induced objects simply as their limit for $r \to \infty$, so that the induced metric or scalar field would be

$$ {}^{\infty}g_{\mu\nu} \stackrel{\text{def}}{=} \lim_{r\to\infty} g_{\mu\nu}, \tag{2.6a} $$

$$ {}^{\infty}\eta \stackrel{\text{def}}{=} \lim_{r\to\infty} \eta. \tag{2.6b} $$

But such a definition would soon prove to be unuseful, for vanishing or infinite limits may appear: for example, for Minkowski space we should have:

$$ {}^{\infty}g_{\mu\nu} \equiv \mathrm{diag}(-1, +1, \dots), \tag{2.7} $$

but for anti-de Sitter we should have, among the various metric components:

$$ {}^{\infty}g_{tt} \equiv \infty, \qquad {}^{\infty}g_{rr} \equiv 0. \tag{2.8} $$

We have a better definition if we use an r-power series expansion around infinity:

$$ {}^{\infty}g_{\mu\nu} \stackrel{\text{def}}{=} {}^{(\alpha)}g_{\mu\nu} + O\left(\frac{1}{r^{\alpha}}\right), \tag{2.9a} $$

$$ {}^{\infty}\eta \stackrel{\text{def}}{=} {}^{(\beta)}\eta + O\left(\frac{1}{r^{\beta}}\right), \tag{2.9b} $$

where α and β are exponents which depend on the metric component $(\mu\nu)$ and on the scalar field respectively, and ${}^{(\alpha)}g_{\mu\nu}$ and ${}^{(\beta)}\eta$ are the metric component $g_{\mu\nu}$ and the scalar field η, expanded in series till the $(1/r^{\alpha-1})$-th and the $(1/r^{\beta-1})$-th powers, respectively. By means of a power series expansion, we can tell what is relevant to the 'surface at infinity' from what is not, while avoiding problems with infinite or vanishing limits: the metric's 'piece' ${}^{(\alpha)}g_{\mu\nu}$ is considered as the effective 'induced metric at infinity', and $O(1/r^{\alpha})$ represents a gauge part; this can in turn be decomposed into an *improper gauge* (lower order terms) and a *proper gauge* (higher order terms, which vanish faster). The distinction between improper and proper gauge is made because the gravitational system's effective physical information resides both in the induced metric at infinity ${}^{(\alpha)}g_{\mu\nu}$ and in the improper gauge part (q.v. Benguria, Cordero, e Teitelboim [9]).

Notice that there appears to be a certain flexibility in choosing the effective metric, improper gauge, and proper gauge parts of the power series expansion; we mean such a choice when we speak about specifying *asymptotic conditions*. The choice is not arbitrary though, but relies rather on physical and mathematical grounds. Consider e.g. the (rr) component of the anti-de Sitter metric in two (1.48a) or three dimensions (1.46 with $A = 0$):

$$ g_{rr} = (\lambda^2 r^2 + \alpha^2)^{-1} = \frac{1}{\lambda^2 r^2} - \frac{\alpha^2}{\lambda^4 r^4} + \frac{\alpha^4}{\lambda^6 r^6} + \cdots ; \tag{2.10} $$

one finds that the most convenient choice is taking $1/(\lambda^2 r^2) + 0/(r^3)$ as the effective part, $-\alpha^2/(\lambda^4 r^4)$ as the improper gauge part, and $O(1/r^5)$ as the proper gauge part: such a choice is dictated by the requirement of having the largest group of asymptotic symmetries with non-diverging charges (q.v. following sections); instead, taking e.g. $1/(\lambda^2 r^2)$ as the effective part and $0/(r^3) - \alpha^2/(\lambda^4 r^4)$ as the improper gauge part would yield diverging charges.[2]

The flexibility in choosing the asymptotic conditions can be viewed as a strong point of this formalism, but we interpret it rather as a sign of ambiguity, since it often happens that one is able to make the right choice only *a posteriori*.

2.1.2 Asymptotic symmetries

The difference between bulk- and surface-symmetries

Given a system whatsoever, it is sometimes possible to deform its bulk and leave its surface unaltered, while, for continuity reasons, it is never possible to do the inverse. Hence, if a certain transformation is *not* a symmetry of the surface (i.e. the surface is deformed by the transformation), it cannot be a symmetry of the bulk either. Therefore one infers that the number of surface symmetries is always equal to or greater than the number of bulk symmetries.

Symmetries at infinity

Once the notions of surface at infinity and metric induced at infinity have been explained, the notion of symmetry at infinity follows thence more or less plainly.

Consider the (left) action of a (Lie) transformation group $\mathcal{G}$ upon a (pseudo-)Riemannian manifold $\mathcal{M}$; a transformation $\mathbf{T}$ belonging to the group is generated by the vector field ξ. The metric $g_{\mu\nu}$ is invariant under the transformation only if its Lie derivative with respect to the generator vanishes:

$$\mathbf{T}(g_{\mu\nu}) = g_{\mu\nu} \Longleftrightarrow \mathbf{L}_\xi g_{\mu\nu} = 0; \tag{2.11}$$

when this happens the transformation $\mathbf{T}$ is an isometry. This is an example of symmetry 'at finite'.

Now consider a surface at infinity with induced metric

$${}^{\infty}g_{\mu\nu} = {}^{(\alpha)}g_{\mu\nu} + O\left(\frac{1}{r^\alpha}\right), \tag{2.12}$$

with specified asymptotic conditions. What shall we mean by symmetry in this case? It is the part ${}^{(\alpha)}g_{\mu\nu}$ that is considered as the effective induced metric, so every transformation that leaves it invariant is to be called a symmetry:

$$\begin{aligned} \mathbf{T} \text{ is a symmetry} &\Longleftrightarrow \mathbf{T}({}^{\infty}g_{\mu\nu}) = {}^{\infty}g_{\mu\nu} \\ &\Longleftrightarrow \mathbf{T}\big({}^{(\alpha)}g_{\mu\nu}\big) = {}^{(\alpha)}g_{\mu\nu} + O\left(\frac{1}{r^\alpha}\right), \end{aligned} \tag{2.13a}$$

[2]Note that the proper gauge part can never be $O(1/r^4)$, otherwise all physical information, i.e. the mass parameter α, would be completely relegated to it.

or, in terms of the generator:

$$\begin{aligned}\mathbf{T} \text{ is a symmetry} \Longleftrightarrow & \ \mathbf{L}_\xi\, {}^{\infty}g_{\mu\nu} = 0 \\ \Longleftrightarrow & \ \mathbf{L}_\xi\, {}^{(\alpha)}g_{\mu\nu} = O\left(\frac{1}{r^\alpha}\right).\end{aligned} \tag{2.13b}$$

Since the generator ξ is just a vector field whose components are functions of the coordinates, it is natural to expand it in r-power series around infinity, and we see from Eq. (2.13b) that its expression is determined but for $O(1/r^\gamma)$ terms:

$$\xi = {}^{(\gamma)}\xi^\mu + O\left(\frac{1}{r^\gamma}\right). \tag{2.14}$$

The ${}^{(\gamma)}\xi^\mu$ (which is determined up to $(1/r^{\gamma-1})$-order terms) represents the effective generator; it generates *improper gauge transformations*, i.e. transformations that leave the effective induced metric invariant but do change the metric's improper gauge part — thus modifying the system's state. The $O(1/r^\gamma)$ term generates *proper gauge transformations* which modify the metric's proper gauge part and thus do not modify the system state (q.v. Benguria, Cordero and Teitelboim [9]).

It is now evident that bulk symmetries are just a subgroup of the asymptotic symmetries; the latter can even be an infinite-dimensional group, as we shall see e.g. in the case of anti-de Sitter space.

Till now, we have spoken only about symmetries which leave the metric field invariant; however, we can consider other fields on the manifold $\mathcal{M}$, like the dilaton η whose invariance is as much important as the metric's; in this case, the symmetry $\mathbf{T}$ must satisfy also the following condition:

$$\mathbf{T}({}^{\infty}\eta) = {}^{\infty}\eta \Longleftrightarrow \mathbf{T}\big({}^{(\beta)}\eta\big) = {}^{(\beta)}\eta + O\left(\frac{1}{r^\beta}\right), \tag{2.15a}$$

or

$$\mathbf{L}_\xi\, {}^{\infty}\eta = 0 \Longleftrightarrow \mathbf{L}_\xi\, {}^{(\beta)}\eta = O\left(\frac{1}{r^\beta}\right). \tag{2.15b}$$

This additional requirement can lead to a reduction in the number of the initial (metric) symmetries (we shall call this 'symmetry breaking'). Anyway, if one is interested in having a symmetry group as large as possible, one can disregard condition (2.15) and consider only condition (2.13); of course, this is only feasible as long as unacceptable physical consequences do not arise, such as diverging charges.

The Holographic Principle

We have seen that the symmetries of a gravitational system can always be tracked back to surface symmetries; this follows simply from Noether's Theorem:

let

$$\mathfrak{L} = \int_{\mathcal{M}} dV \sqrt{-\boldsymbol{g}}\, L \tag{2.16}$$

be the action of a gravitational system model; it is invariant under a transformation $\mathbf{T}_\xi$ if and only if its Lie derivative with respect to the transformation generator ξ vanishes:

$$\mathbf{L}_\xi \mathfrak{L} = \int_{\mathcal{M}} dV \mathbf{L}_\xi (\sqrt{-\boldsymbol{g}}\, L) = 0. \tag{2.17}$$

Since $\sqrt{-\boldsymbol{g}}\, L$ is a scalar density, and the Lie derivative of a scalar density $\boldsymbol{s}$ can be expressed as $\mathbf{L}_\xi \boldsymbol{s} \equiv \partial_\mu (\xi^\mu \boldsymbol{s})$, the preceding equation becomes:

$$\mathbf{L}_\xi \mathfrak{L} \equiv \int_{\mathcal{M}} dV \partial_\mu (\xi^\mu \sqrt{-\boldsymbol{g}}\, L) = 0, \tag{2.18}$$

or, by Stokes' Theorem:

$$\mathbf{L}_\xi \mathfrak{L} \equiv \int_{\partial\mathcal{M}} ds_\mu \xi^\mu \sqrt{-\boldsymbol{g}}\, L = 0. \tag{2.19}$$

Hence the invariance of a general gravitational action is only determined by the asymptotic behaviour of the latter.

However, one may ask whether this is just a consequence of Noether's Theorem, or of something deeper. For example, the fact that the invariance is shifted from a (D+1)-dimensional context to a D-dimensional one could be a signal of a correspondence between two theories in different dimensions. Such a conjecture is just the Holographic Principle as formulated by Susskind [43], about which we spoke in the Introduction (q.v. also Aharony, Gubser, Maldacena, Ooguri, Oz [2]).

2.1.3 Conserved charges associated to the asymptotic symmetries

A classical relativistic example

A system's invariance under a group of transformations gives rise to conserved charges (Noether's Theorem); a plain example from General Relativity is the following: consider the equation for the matter stress-energy tensor:

$$\boldsymbol{\nabla}_\mu T^{\mu\nu} = 0; \tag{2.20}$$

now suppose there is an isometry, generated by the Killing vector field ξ^μ which satisfies

$$\mathbf{L}_\xi g_{\mu\nu} \equiv \boldsymbol{\nabla}_{(\mu} \xi_{\nu)} = 0; \tag{2.21}$$

contract Eq. (2.20) with the Killing vector field: from Eq. (2.21) and from $T^{\mu\nu}$'s being symmetric it follows that:

$$\boldsymbol{\nabla}_\mu \xi_\nu T^{\mu\nu} = 0. \tag{2.22}$$

Go on to integrate this equation over a spatially infinite four-dimensional volume $\mathcal{V}$, whose upper and lower boundaries are two arbitrary three-dimensional spacelike hypersurfaces, $\partial\mathcal{V}_{t''}$ and $\partial\mathcal{V}_{t'}$, defined by the equations $t' = \text{const.}$ and $t'' = \text{const.}$:

$$\int_{\mathcal{V}} \sqrt{-\boldsymbol{g}}\, \boldsymbol{\nabla}_\mu \xi_\nu T^{\mu\nu} = 0. \tag{2.23}$$

The integrand in the last equation is a divergence and can be rewritten as a sum of integrals over the boundary of $\mathcal{V}$ by Stokes' Theorem. If the matter stress-energy tensor vanishes at spatial infinity, the sum reduces to:

$$\int_{\partial\mathcal{V}_{t''}} \sqrt{\boldsymbol{h}}\, u_\mu \xi_\nu T^{\mu\nu} - \int_{\partial\mathcal{V}_{t'}} \sqrt{\boldsymbol{h}}\, u_\mu \xi_\nu T^{\mu\nu} = 0; \tag{2.24}$$

since the two hypersurfaces were arbitrary, this means that the charge defined by

$$\mathfrak{C}_t(\xi) \stackrel{\text{def}}{=} \int_{\partial\mathcal{V}_t} \sqrt{\boldsymbol{h}}\, u_\mu \xi_\nu T^{\mu\nu} \tag{2.25}$$

is conserved in time.

In the preceding example it is crucial the fact that a Killing vector field, i.e. a symmetry, exists globally over the whole manifold ('bulk' symmetry). The charges associated to asymptotic symmetries, instead, always require only local properties around infinity; their study and computation can be made by means of the Hamiltonian formalism or by Brown and York's quasilocal formalism: each formalism is related to the other; they will be discussed in detail in the following sections.

2.2 Asymptotic symmetries in the Hamiltonian formalism

2.2.1 Regge and Teitelboim's procedure

The role of asymptotic symmetries became clear with the development of the Hamiltonian formalism for gravity theory; the way their charges are calculated reflects just this development.

The 'first' gravitational Hamiltonian was indeed constructed following about the same steps we took in Sec. 1.2.2, but without accounting for the boundary terms which arise in the various applications of Stokes' Theorem; hence its form was simply:

$$\int_{\mathcal{S}} (N\boldsymbol{H} + N^i \boldsymbol{H}_i); \tag{2.26}$$

of course, the right equations of motion were found all the same, because their form is determined by the volume integral only.

It was noted first by Dirac and DeWitt [26] the fact that the Hamiltonian (2.26) needs a boundary term whose form is

$$\mathfrak{E} \stackrel{\text{def}}{=} \kappa \int_{\mathcal{P}} \sqrt{\boldsymbol{\sigma}}\, \tilde{n}^k (\mathbf{D}_i h_{jk} - \mathbf{D}_k h_{ij}) h^{ij}; \tag{2.27}$$

This integral is (asymptotically) none but the term

$$\mathfrak{E} \equiv \int_{\mathcal{P}} N \boldsymbol{E} \tag{2.28a}$$

with

$$\boldsymbol{E} \stackrel{\text{def}}{=} \sqrt{\boldsymbol{\sigma}}\, u_\mu u_\nu 2\Pi^{\mu\nu} - \underline{\boldsymbol{E}}, \tag{2.28b}$$

$$\underline{\boldsymbol{E}} \stackrel{\text{def}}{=} \sqrt{\boldsymbol{\sigma}}\, u_\mu u_\nu 2\underline{\Pi}^{\mu\nu}, \tag{2.28c}$$

that we have already seen in Sec. 1.2.2. DeWitt noticed that the gravitational Hamiltonian would not yield linearized gravity theory and that there would not be any definition of energy without the additional surface term. In fact it is *the* energy, because the Hamiltonian volume integral vanish identically by virtue of the constraints $\boldsymbol{H} = 0$ and $\boldsymbol{H}^i = 0$. From this point of view, quoting DeWitt, gravity theory is unique among field theories in that its energy may always be expressed as a surface integral. Therefore, the energy depends upon the *asymptotic characteristics* of the gravitational system. This fact is due to the theory's being diffeomorphism-invariant: many of its degrees of freedom are unphysical, and maybe the physical ones can be sought for in the boundary.

Regge and Teitelboim [39] were the first to give a formal and physical justification for the presence of a surface term in the gravitational Hamiltonian; they showed that, without such term, the action would be defined in a phase-space laking the trajectories that should extremise it. In fact, suppose to strip the action (1.15) (without dilaton for simplicity) of its boundary terms and to use it for the variational principle: the variation would be:

$$\begin{aligned} \delta\mathfrak{H}^1 = {} & \int_{\mathcal{S}} (\boldsymbol{A}^{ij} \delta h_{ij} + B_{ij} \delta \boldsymbol{P}^{ij}) \\ & - \int_{\mathcal{P}} \frac{\sqrt{\boldsymbol{\sigma}}}{\sqrt{\boldsymbol{h}}} \tilde{n}_l [\boldsymbol{G}^{ijkl} (N \mathbf{D}_k \delta h_{ij} - \partial_k N\, \delta h_{ij}) \\ & + 2N_i\, \delta \boldsymbol{P}^{il} + (2N^i \boldsymbol{P}^{kl} - N^l \boldsymbol{P}^{ik}\, \delta h_{ik})], \end{aligned} \tag{2.29a}$$

with

$$\boldsymbol{G}^{ijkl} \stackrel{\text{def}}{=} \frac{1}{2} \sqrt{\boldsymbol{h}}\, (h^{ik} h^{jl} + h^{il} h^{jk} - 2 h^{ij} h^{kl}). \tag{2.29b}$$

In order to obtain from the variation above the canonical equations of motion

$$\dot{h}_{ij} = \frac{\delta \mathfrak{H}^1}{\delta \boldsymbol{P}^{ij}}, \tag{2.30a}$$

$$\dot{\boldsymbol{P}}^{ij} = -\frac{\delta \mathfrak{H}^1}{\delta h_{ij}}, \tag{2.30b}$$

it would be necessary for the surface integrals to vanish. But consider a physically reasonable solution such as Schwarzschild's,

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2, \tag{2.31}$$

which behaves asymptotically like (in a Cartesian coordinate system):

$$g_{tt} = -1 + O\left(\frac{1}{r}\right), \tag{2.32}$$

$$g_{ij} = \delta_{ij} + O\left(\frac{1}{r}\right); \tag{2.33}$$

one can easily convince oneself that Schwarzschild's solution does *not* make the surface integrals in (2.29a) vanish, hence it should not belong to the phase-space wherein the action — stripped from its boundary terms — is well defined.

The surface integral (2.27) does just this: it makes all anomalous surface terms in (2.29a) vanish, thus redefining the phase-space in a physically more appropriate manner, enlarging it to contain physically quite reasonable solutions. (*En passant*, we wish to mark that the counterterm $\underset{\sim}{\boldsymbol{E}}$ in (2.28) is absolutely necessary for such redefinition.)

The anomalous non-vanishing boundary terms are strictly related to the *asymptotic conditions* (2.32): different asymptotic conditions can make other non-vanishing boundary terms appear, so that one would need other additional surface terms in the original Hamiltonian besides (2.27). Hence, the general method is the following:

1. calculate the variation of the (bulk) Hamiltonian, thus obtaining the boundary terms;

2. fix the asymptotic conditions of the class of solutions that are to belong to phase-space (generally all members of this class are just like 'excitations' of the same 'ground state' solution);

3. examine which surface integrals in the variation are asymptotically non-vanishing for the fixed asymptotic conditions;

4. integrate the non-vanishing variational integrals so as to have finite terms which are then subtracted from the original Hamiltonian.

This way the variational principle will be well defined. It should be noted that steps 2. and 4. involve an implicit choice of a reference, or background, spacetime (the 'ground state', related to the term $\underset{\sim}{\boldsymbol{E}}$). One could also choose to retain *all* (integrated) surface integrals, rather than the non-vanishing ones only; both choices yield a well-defined Hamiltonian anyway, and the same results. An important point is that it is *not always* possible to integrate the variations so as to have the appropriate integrals to be subtracted from the Hamiltonian. We shall see some examples of this problems in the next chapter.

The method outlined above has been used by Regge and Teitelboim [39] with Minkowski space as reference spacetime, by Brown and Henneaux [14] with three-dimensional anti-de Sitter space as reference spacetime, and by Cadoni and Mignemi [19] for dilatonic gravity on two-dimensional anti-de Sitter space; in the present work, we shall adopt this method (but not exclusively) for dilatonic gravity on three-dimensional anti-de Sitter space.

We have already seen how the specification of asymptotic conditions determines a group of asymptotic symmetries. If one lets the Hamiltonian evolve under an asymptotic-symmetry generator (introducing the latter in the lapse and shift as explained in Sec. 1.2.3), then the boundary term will give the associated charge (q.v. Benguria, Cordero, Teitelboim [9]): the bulk term of the Hamiltonian gives no contribute thereto, since the constraints $\boldsymbol{H}$ and $\boldsymbol{H}_i$ vanish; this fact clearly means that the system is indifferent to the action of the generator upon its bulk.

2.2.2 Recently adopted Hamiltonian surface terms

In the most recent papers on the gravitational Hamiltonian, all steps seen in Sec. 1.2.2 are usually taken in its derivation, and all boundary terms are retained, so that a general Hamiltonian appears as:

$$\mathfrak{H} \stackrel{\text{def}}{=} \int_{\mathcal{S}} (N\boldsymbol{H} + N^i\boldsymbol{H}_i) + \int_{\mathcal{P}} (\tilde{N}\boldsymbol{E} - \tilde{N}^A\boldsymbol{J}_A), \tag{2.34}$$

where all the information about the charges is thus associated with the boundary integral

$$\begin{aligned}\int_{\mathcal{P}} (\tilde{N}\boldsymbol{E} - \tilde{N}^A\boldsymbol{J}_A) = \int_{\mathcal{P}} \sqrt{\boldsymbol{\sigma}}\,[\tilde{N}(-2n^\mu \boldsymbol{\nabla}_\mu \eta + 2\eta k - \underset{\sim}{E})& \\ - \tilde{N}^A(2\sqrt{\boldsymbol{\sigma}}\,\gamma_{Ai}n_jP^{ij} - \underset{\sim}{J}_A)],&\end{aligned} \tag{2.35}$$

with $\tilde{N}\tilde{u}^\mu + \tilde{N}^\mu = Nu^\mu + N^\mu$. Since the Hamiltonian above should be well-defined, one infers that its boundary integral (2.35) should be equivalent to the following:

$$\begin{aligned}\int_{\mathcal{P}} \frac{\sqrt{\boldsymbol{\sigma}}}{\sqrt{\boldsymbol{h}}} \tilde{n}_l \{ \kappa \boldsymbol{G}^{ijkl} &[N\eta \mathbf{D}_k h_{ij} - \partial_k(N\eta)(h_{ij} - \underline{h}_{ij})] \\ &+ \kappa \sqrt{\boldsymbol{h}}\,(2h^{il}h^{jk} - h^{ij}h^{kl})(h_{ij} - \underline{h}_{ij}) N\,\partial_k \eta \\ &+ 2\kappa \sqrt{\boldsymbol{h}}\,[\partial_j N(\eta - \underline{\eta}) - N\partial_j(\eta - \underline{\eta})] \\ &\quad + (2N^i \boldsymbol{P}^{kl} - N^l \boldsymbol{P}^{ik})(h_{ik} - \underline{h}_{ik}) \\ &\quad + 2N_i(\boldsymbol{P}^{il} - \underline{\boldsymbol{P}}^{il}) - N^l \boldsymbol{P}^{\boldsymbol{\eta}}(\eta - \underline{\eta})\},\end{aligned} \tag{2.36}$$

which is just the one derived after Regge and Teitelboim's method. Who writes has not verified this presumed equivalence. We leave this important question aside by now; we shall use the term (2.35) to calculate the charge, and only then shall we draw some conclusions, comparing the results with the ones obtained by the term *à la* Regge and Teitelboim.

The surface term (2.35) is not the only one to have appeared in the literature; another one is e.g. Hawking and Hunter's [30]):

$$\int_{\mathcal{P}} (N\boldsymbol{E} - N^i \boldsymbol{J}_i) \tag{2.37a}$$

with

$$\boldsymbol{E} \stackrel{\text{def}}{=} \sqrt{\boldsymbol{\sigma}} \left(2k - 2\frac{\alpha}{\cosh\alpha} \boldsymbol{\nabla}_\mu \tilde{u}^\mu \right) - \underline{\boldsymbol{E}}, \tag{2.37b}$$

$$\boldsymbol{J}_i \stackrel{\text{def}}{=} 2\sqrt{\boldsymbol{\sigma}}\, \tilde{n}^j P_{ji} - \underline{\boldsymbol{J}}_i, \tag{2.37c}$$

which we have already seen in Sec. 1.2.4.

2.3 Asymptotic symmetries in the quasilocal formalism of Brown and York

2.3.1 Brown and York's quasilocal stress-energy tensor

Definition

Brown and York [17] have developed a formalism for the definition of a quasilocal energy and conserved charges of a gravitational system, by means of an analogy with the Hamilton-Jacobi classical formalism for point dynamics.

They consider the classical action which describes the unidimensional motion of a point particle:

$$S^1 = \int dt \left(p\frac{dx}{dt} - H^1(x,p,t) \right); \tag{2.38}$$

by parameterizing the system's trajectory in phase-space with the parameter λ, the action takes the form:

$$S^1 = \int_{\lambda'}^{\lambda''} d\lambda \left(p\frac{dx}{d\lambda} - \frac{dt}{d\lambda} H^1(x,p,t) \right); \tag{2.39}$$

and its variation is:

$$\begin{aligned} \delta S^1 &= [\text{terms giving the equations of motion}] \\ &\quad + p\,\delta x|_{\lambda'}^{\lambda''} - H^1\,\delta t|_{\lambda'}^{\lambda''}. \end{aligned} \tag{2.40}$$

The last two terms in the variation vanish since the action's domain is the space of trajectories having fixed-valued end-points, hence the action is extremised only if the equations of motion hold. Instead, if we take as domain the space of trajectories satisfying the equations of motion but having free-valued end-points, the first term in the variation (2.40) vanish, so that

$$\delta S^1_{\text{cl}} = p_{\text{cl}}\,\delta x|_{\lambda'}^{\lambda''} - H^1_{\text{cl}}\,\delta t|_{\lambda'}^{\lambda''}, \tag{2.41}$$

where 'cl' denotes evaluation in the new domain.

From Eq. (2.41) we have the Hamilton-Jacobi equations

$$p_{\text{cl}}|_{\lambda''} = \frac{\partial S^1_{\text{cl}}}{\partial x''}, \tag{2.42a}$$

$$H^1_{\text{cl}}|_{\lambda''} = -\frac{\partial S^1_{\text{cl}}}{\partial t''}, \tag{2.42b}$$

where $x'' = x(\lambda'')$ and $t'' = t(\lambda'')$. Eq. (2.42b) defines the system's energy at point λ'', and it is taken by Brown and York as the starting point for the construction of a quasilocal energy from the gravitational action.

Consider the action (1.9) restricted to the non-dilatonic case:

$$\begin{aligned} \mathfrak{L} \overset{\text{def}}{=} \kappa \int_{\mathcal{M}} \sqrt{-\boldsymbol{g}}\, R_{\mathcal{M}} + 2\kappa \int_{\mathcal{S}'}^{\mathcal{S}''} \sqrt{\boldsymbol{h}}\, K \\ - 2\kappa \int_{\mathcal{B}} \sqrt{-\gamma}\,\Theta + 2\kappa \int_{\mathcal{P}'}^{\mathcal{P}''} \sqrt{\boldsymbol{\sigma}}\,\alpha + \underline{\mathfrak{L}} + \mathfrak{L}^{\text{mat}}; \end{aligned} \tag{2.43}$$

the domain is the space of the metrics which induce a fixed D-dimensional metric on the boundary $\partial\mathcal{M}$; in fact, upon examining the variation:

$$\begin{aligned} \delta\mathfrak{L} = -\kappa \int_{\mathcal{M}} \sqrt{-\boldsymbol{g}}\, G^{\mu\nu}\delta g_{\mu\nu} + \int_{\mathcal{S}'}^{\mathcal{S}''} \boldsymbol{P}^{\mu\nu}\delta h_{\mu\nu} + \int_{\mathcal{B}} \boldsymbol{\Pi}^{\mu\nu}\delta\gamma_{\mu\nu} \\ + \int_{\mathcal{P}'}^{\mathcal{P}''} \boldsymbol{\pi}^{\mu\nu}\delta\sigma_{\mu\nu} + \int_{\mathcal{B}} \underline{\boldsymbol{\Pi}}^{ab}\delta\gamma_{ab} + \tfrac{1}{2}\int_{\mathcal{M}} \sqrt{-\boldsymbol{g}}\, T^{\mu\nu}\delta g_{ab}, \end{aligned} \tag{2.44}$$

since $\delta h_{ij} = \delta\gamma_{ab} = \delta\sigma_{AB} \equiv 0$, the action is extremised if the equations of motion $G^{\mu\nu} = \frac{1}{2\kappa}T^{\mu\nu}$ hold.

However, also in this case we can consider a new domain, i.e. the space of the metrics on $\mathcal{M}$ which satisfy the equations of motion and are not fixed on the boundary; in analogy with Eq. (2.41), the action's variation now reduces to:

$$\begin{aligned}\delta\mathfrak{L}_{\rm cl} = &\int_{\mathcal{S}'}^{\mathcal{S}''} \boldsymbol{P}_{\rm cl}{}^{ij}\delta h_{ij} + \int_{\mathcal{P}'}^{\mathcal{P}''} \boldsymbol{\pi}_{\rm cl}{}^{ij}\delta\sigma_{ij} \\ &+ \int_{\mathcal{B}} \boldsymbol{\Pi}_{\rm cl}{}^{ab}\delta\gamma_{ab} + \int_{\mathcal{B}} \underline{\boldsymbol{\Pi}}^{ab}\delta\gamma_{ab},\end{aligned} \tag{2.45}$$

where 'cl' again denotes the new domain.

If we want to look for an expression that should be the analogue of Hamilton-Jacobi equation (2.42b), we must first note that, in the gravitational case, the quantity on the boundary is not just the elapsed time $(t'' - t')$ as it is in Eq. (2.42b), but it is a metric γ_{ab} which determines every timelike and space-like interval in the manifold $\mathcal{B}$. The last consideration leads Brown and York to the definition of a surface stress-energy tensor associated to $\mathcal{B}$:

$$\tau^{ab} \stackrel{\rm def}{=} \frac{2}{\sqrt{-\gamma}}\frac{\delta\mathfrak{L}_{\rm cl}}{\delta\gamma_{ab}} = \frac{2}{\sqrt{-\gamma}}(\boldsymbol{\Pi}_{\rm cl}{}^{ab} - \underline{\boldsymbol{\Pi}}^{ab}) = 2(\Pi_{\rm cl}{}^{ab} - \underline{\Pi}^{ab}). \tag{2.46}$$

In the above definition, it is important to note that the matter term $\mathfrak{L}^{\rm mat}$ as well contributes to the tensor τ^{ab}, which thus characterizes the whole gravity-matter system. That does not happen with the usual stress-energy tensor

$$T^{\mu\nu} \equiv \frac{2}{\sqrt{-\boldsymbol{g}}}\frac{\delta\mathfrak{L}^{\rm mat}}{\delta g_{\mu\nu}}, \tag{2.47}$$

which characterizes only the system's matter fields.

Additive counterterms

The term $\underline{\mathfrak{L}}[\gamma_{ab}]$ in Eq. (2.43), which depends only on the metric induced on $\mathcal{B}$, is the analogue of an arbitrary function S^0 subtracted from the classical action (2.38), which depends only on the coordinates of the end-points of the system's trajectory:

$$S[x,p] = S^1[x,p] - S^0[x', x'', t', t'']; \tag{2.48}$$

this function does not contribute to the equations of motion since its variation vanishes by virtue of the end-point value fixation; however, its presence does shift the definition of the zero-point energy:

$$H_{\rm cl}|_{\lambda''} = H^1_{\rm cl}|_{\lambda''} - H^0|_{\lambda''} = -\left(\frac{\partial S^1_{\rm cl}}{\partial t''} - \frac{\partial S^0}{\partial t''}\right). \tag{2.49}$$

In an analogous way, the functional $\underline{\mathfrak{L}}$ does not alter the equations of motion, but does intervenes in the definition of the momentum conjugate to γ_{ab} and hence in the definition of the quasilocal stress-energy tensor, as Eq. (2.46) plainly shows.

We can find some requirement to be satisfied by the counterterm, by reasoning as follows: the various tangential and parallel projections of the quasilocal tensor with respect to the boundary $\mathcal{B}$ give the energy, momentum, and stress densities:

$$\boldsymbol{E} \stackrel{\text{def}}{=} \sqrt{\boldsymbol{\sigma}}\, \tilde{u}_a \tilde{u}_b \tau^{ab} = -\frac{\delta \mathfrak{L}_{\text{cl}}}{\delta N}, \tag{2.50a}$$

$$\boldsymbol{J}_A \stackrel{\text{def}}{=} -\sqrt{\boldsymbol{\sigma}}\, \sigma_{Aa} \tilde{u}_b \tau^{ab} = \frac{\delta \mathfrak{L}_{\text{cl}}}{\delta N^A}, \tag{2.50b}$$

$$\boldsymbol{s}^{AB} \stackrel{\text{def}}{=} \sqrt{-\gamma}\, \sigma^A{}_a \sigma^B{}_b \tau^{ab} = 2 \frac{\delta \mathfrak{L}_{\text{cl}}}{\delta \sigma_{AB}}; \tag{2.50c}$$

their explicit form is (note that no dilaton field is considered at the moment):

$$\boldsymbol{E} = 2\kappa \sqrt{\boldsymbol{\sigma}}\, k - \utilde{\boldsymbol{E}}, \tag{2.51a}$$

$$\boldsymbol{J}_A = -2\sqrt{\boldsymbol{\sigma}}\, \gamma_{Ai} n_j P^{ij} - \utilde{\boldsymbol{J}}_A, \tag{2.51b}$$

$$\boldsymbol{s}^{AB} = 2\kappa \sqrt{\boldsymbol{\sigma}}\, [k^{AB} + \sigma^{AB}(n^\mu u^\nu \boldsymbol{\nabla}_\nu u_\mu - k)] - \utilde{\boldsymbol{s}}^{AB}. \tag{2.51c}$$

The quantities $\boldsymbol{E}$ and $\boldsymbol{J}_A$ are the same as those in the surface term (2.35); we have already mentioned the fact that they depend exclusively on the canonical variables only if the quantities $\utilde{\boldsymbol{E}}$ and $\utilde{\boldsymbol{J}}_A$ do. For this to happen, it is necessary that the counterterm $\utilde{\mathfrak{L}}$ be a *linear* functional of the (boundary) lapse and shift:

$$\utilde{\mathfrak{L}} \stackrel{\text{def}}{=} \int_{\mathcal{B}} (\tilde{N} \utilde{\boldsymbol{E}} - \tilde{N}^A \utilde{\boldsymbol{J}}_A). \tag{2.52}$$

Definition with respect to a reference spacetime A way to define the counterterm $\utilde{\mathfrak{L}}$ with respect to a reference spacetime ('ground state') is the following: one chooses a particular solution $(\utilde{\mathcal{M}}, \utilde{g}_{\mu\nu})$ of the equations of motion (preferably one with nice properties such as staticity), to be considered as a ground state, and isometrically embeds the boundary $\mathcal{B}$ therein; the quantities $\utilde{\boldsymbol{E}}$ and $\utilde{\boldsymbol{J}}_A$ are then defined as the energy and the momentum as calculated in the embedding; hence it is obvious that $\boldsymbol{E}$ and $\boldsymbol{J}_A$ will vanish for the ground state.

As a consequence of this definition of $\utilde{\mathfrak{L}}$, the term $\utilde{\boldsymbol{s}}^{AB}$ is defined as

$$\utilde{\boldsymbol{s}}^{AB} \stackrel{\text{def}}{=} 2 \frac{\delta \utilde{\mathfrak{L}}}{\delta \sigma_{AB}} = 2\tilde{N} \frac{\delta \utilde{\boldsymbol{E}}}{\delta \sigma_{AB}} + \tilde{N}^C \frac{\delta \utilde{\boldsymbol{J}}_C}{\delta \sigma_{AB}}, \tag{2.53}$$

which follows from the variation

$$\delta \utilde{\mathfrak{L}} = \int_{\mathcal{B}} \left[\utilde{\boldsymbol{E}} \delta \tilde{N} + \utilde{\boldsymbol{J}}_A \delta \tilde{N}^A + \left(\tilde{N} \frac{\delta \utilde{\boldsymbol{E}}}{\delta \sigma_{AB}} + \tilde{N}^A \frac{\delta \utilde{\boldsymbol{J}}_A}{\delta \sigma_{AB}} \right) \delta \sigma_{AB} \right]; \tag{2.54}$$

such a definition for $\utilde{\boldsymbol{s}}^{AB}$ is *not* equivalent to the definition with respect to the reference spacetime — as opposed to $\utilde{\boldsymbol{E}}$ and $\utilde{\boldsymbol{J}}_A$'s definitions.

This method of defining the counterterm $\utilde{\mathfrak{L}}$ is physically reasonable, but presents two problems: first, the choice of the reference spacetime; and second, the fact that the isometrical embedding may happens not to be unique or even not to exist. Anyway, such an embedding does exist and is unique in important examples like Minkowski or anti-de Sitter space; hence many authors, besides Brown and York, like Hawking and Horowitz [29], Hawking and Hunter [30], Booth and Mann [11, 12], Brown, Creighton and Mann [15], Brown, Lau and York [16], Bose and Dadhich [13], use this counterterm definition[3]

Intrinsic definition Balasubramanian and Kraus [5] have recently proposed an alternative way of specifying the surface term $\underline{\mathfrak{L}}$, which uses the main intrinsic metric objects of $\mathcal{B}$ (like metric, volume element, scalar curvature, Ricci and Riemann tensors, etc.) and the requirement that the system's charges be asymptotically divergenceless. This method is geometrically reasonable (no reference-spacetime choices or embedding problems), but yields anomalous results sometimes (e.g., the energy inside an ellipsoidal surface in Minkowski space is different from that inside a sphere, which obviously vanishes). However, as has been noticed by Lau [33], the 'intrinsic' method and the 'background' one are asymptotically equivalent. We shall make use of this equivalence in some of the calculations to follow.

The 'intrinsic' method is used, besides Balasubramanian and Kraus [5] and Lau [33], also by Mann [36], and by Emparan, Johnson and Myers [28].

Equation of motion for the quasilocal tensor and conserved charges

The similarity between the definitions of the matter tensor $T^{\mu\nu}$ and of the quasilocal one τ^{ab} extends to a similarity in their equations of motion.

In fact, the requirement that the actions $\mathfrak{L}$ and $\mathfrak{L}^{\mathrm{mat}}$ be (separately) invariant under diffeomorphisms yields the $\mathcal{B}$-boundary constraint equation[4]:

$$2\mathbf{\Delta}_b {\Pi_{\mathrm{cl}}}^{ab} = -\gamma^a{}_\mu n_\nu T^{\mu\nu}, \tag{2.55}$$

whence, substituting the definition (2.46), we have the equation of motion for the quasilocal tensor:

$$\mathbf{\Delta}_b \tau^{ab} = -\gamma^a_\mu n_\nu T^{\mu\nu}. \tag{2.56}$$

The equation above differs from the usual energy-stress tensor's one ($\mathbf{\nabla}_\nu T^{\nu\mu} = 0$) by the presence of a source term.

Now, suppose that $\mathcal{B}$ possesses a Killing vector field ξ^a:

$$\mathbf{L}_\xi \gamma_{ab} \equiv \mathbf{\Delta}_{(a}\xi_{b)} = 0; \tag{2.57}$$

contracting Eq. (2.56) with ξ^a, using Eq. (2.57), and integrating between two arbitrary spacelike surfaces $\mathcal{P}_{t'}$ and $\mathcal{P}_{t''}$ in $\mathcal{B}$, one has:

$$\int_{t'}^{t''} \sqrt{-\gamma}\, \xi^a \mathbf{\Delta}_b {\tau_a}^b = -\int_{\mathcal{P}_{t'}}^{\mathcal{P}_{t''}} \sqrt{-\gamma}\, \xi^a \gamma_{a\mu} n_\nu T^{\mu\nu}, \tag{2.58}$$

[3] In fact, the definitions given by some of the listed authors differ slightly from the one presented here; but all the listed authors' definitions use a background spacetime.

[4] Note that this equation is *not* equivalent to the Einstein equation with one index projected normally, and the other index projected tangentially to $\mathcal{B}$, as Brown and York [17] say.

which, by Stokes' Theorem, is just:

$$-\int_{\mathcal{P}_t''}\sqrt{\boldsymbol{\sigma}}\,\xi^a\tilde{u}^b\tau_{ab}+\int_{\mathcal{P}_t'}\sqrt{\boldsymbol{\sigma}}\,\xi^a\tilde{u}^b\tau_{ab}=-\int_{t'}^{t''}\sqrt{-\boldsymbol{\gamma}}\,\xi^a\gamma_{\mu a}n_\nu T^{\mu\nu}; \quad (2.59)$$

hence we find Brown and York's equation for the charge associated to the Killing vector ξ^a; it can be written as:

$$\mathfrak{Q}_{\mathcal{P}_{t''}}(\xi)-\mathfrak{Q}_{\mathcal{P}_{t'}}(\xi)=\int_{\mathcal{P}_{t'}}^{\mathcal{P}_{t''}}\sqrt{-\boldsymbol{\gamma}}\,\xi^a\gamma_{a\mu}n_\nu T^{\mu\nu}, \quad (2.60)$$

where

$$\mathfrak{Q}_{\mathcal{P}_t}(\xi)\stackrel{\text{def}}{=}\int_{\mathcal{P}_t}\sqrt{\boldsymbol{\sigma}}\,\xi^a\tilde{u}^b\tau_{aa} \quad (2.61)$$

is the charge associated to the killing vector field ξ^a evaluated at $\mathcal{P}_t$.

One can see that the charge $\mathfrak{Q}_{\mathcal{P}}(\xi)$ does not depend on the choice of a specific surface $\mathcal{P}$ (so that the index '$_\mathcal{P}$' can be omitted) only if, in $\mathcal{B}$, one has $T^{\mu\nu}=0$ (sufficient condition) or $n_\mu T^{\mu\nu}\gamma_{\nu a}\xi^a=0$ (necessary condition); in such cases the charge is constant in time and represents a conserved charge associated to the Killing vector ξ. Note that the charge depends on the normalisation of the Killing vector, indeed it is evident that $\mathfrak{Q}(c\xi)=c\mathfrak{Q}(\xi)$. When the Killing vector is timelike (hence the hypersurface $\mathcal{B}$ is stationary), $\xi\equiv\partial/\partial t$, its charge evaluated at $\mathcal{P}$ is identified with the energy inside whichever spacelike hypersurface $\mathcal{S}$ having $\mathcal{P}$ as its boundary.

Note that the pseudo-vector $\sqrt{\boldsymbol{\sigma}}\,\tilde{u}^b\tau_{ab}$ in Eq. (2.61) can be decomposed as follows by means of Eqs. (2.50):

$$\sqrt{\boldsymbol{\sigma}}\,\tilde{u}^b\tau_{ab}\equiv-\boldsymbol{E}\tilde{u}_a-\boldsymbol{J}_a, \quad (2.62)$$

where $\boldsymbol{E}$ and $\boldsymbol{J}_a$ are just the energy and momentum densities of Eq. (2.35): this implies that the expression for the charge (2.61) is just (minus) the Hamiltonian boundary integral (2.35), with $\xi^a=\tilde{N}\tilde{u}^a+\tilde{N}^a$. It follows that using the quasilocal formalism for calculating the asymptotic-symmetry charges is equivalent to using the Hamiltonian method discussed previously, but with a boundary term like (2.35) instead of a Regge-Teitelboim one.

2.3.2 Quasilocal tensor for a dilaton gravity theory

Definition

Now we wish to extend the definition of the quasilocal stress-energy tensor to the case of a dilaton gravity theory described by the action (1.9):

$$\begin{aligned}\mathfrak{L}\stackrel{\text{def}}{=}\kappa\int_{\mathcal{M}}\sqrt{-\boldsymbol{g}}\,\eta(R_{\mathcal{M}}+\Lambda)+2\kappa\int_{\mathcal{S}'}^{\mathcal{S}''}\sqrt{\boldsymbol{h}}\,\eta K\\-2\kappa\int_{\mathcal{B}}\sqrt{-\boldsymbol{\gamma}}\,\eta\Theta+2\kappa\int_{\mathcal{P}'}^{\mathcal{P}''}\sqrt{\boldsymbol{\sigma}}\,\eta\alpha+\underline{\mathfrak{L}}+\mathfrak{L}^{\text{mat}}.\end{aligned} \quad (2.63)$$

The variation of the action above (omitting the terms coming from the variation of the matter fields) is:

$$
\begin{aligned}
\delta\mathfrak{L} = {} & \int_{\mathcal{M}} (\boldsymbol{\Xi}^{\mu\nu}\delta g_{\mu\nu} + \boldsymbol{\Xi}^{\boldsymbol{\eta}}\delta\eta) + \int_{\mathcal{S}'}^{\mathcal{S}''} (\boldsymbol{P}^{\mu\nu}\delta h_{\mu\nu} + \boldsymbol{P}^{\boldsymbol{\eta}}\delta\eta) \\
& + \int_{\mathcal{B}} (\boldsymbol{\Pi}^{\mu\nu}\delta\gamma_{\mu\nu} + \boldsymbol{\Pi}^{\boldsymbol{\eta}}\delta\eta) + \int_{\mathcal{P}'}^{\mathcal{P}''} (\boldsymbol{\pi}^{\mu\nu}\delta\sigma_{\mu\nu} + \boldsymbol{\pi}^{\boldsymbol{\eta}}\delta\eta) \\
& + \int_{\mathcal{B}} (\underline{\boldsymbol{\Pi}}^{ab}\delta\gamma_{ab} + \underline{\boldsymbol{\Pi}}^{\boldsymbol{\eta}}\delta\eta) + \int_{\mathcal{M}} \tfrac{1}{2}\sqrt{-\boldsymbol{g}}\, T^{\mu\nu}\delta g_{ab};
\end{aligned}
\tag{2.64}
$$

we consider the variation in a configuration-space where the equations of motion are satisfied and the induced boundary metrics are not fixed:

$$
\begin{aligned}
\delta\mathfrak{L}_{\mathrm{cl}} = {} & \int_{\mathcal{S}'}^{\mathcal{S}''} (\boldsymbol{P}_{\mathrm{cl}}{}^{\mu\nu}\delta h_{\mu\nu} + \boldsymbol{P}^{\boldsymbol{\eta}}{}_{\mathrm{cl}}\delta\eta) + \int_{\mathcal{B}} (\boldsymbol{\Pi}_{\mathrm{cl}}{}^{\mu\nu}\delta\gamma_{\mu\nu} + \boldsymbol{\Pi}^{\boldsymbol{\eta}}{}_{\mathrm{cl}}\delta\eta) \\
& + \int_{\mathcal{P}'}^{\mathcal{P}''} (\boldsymbol{\pi}_{\mathrm{cl}}{}^{\mu\nu}\delta\sigma_{\mu\nu} + \boldsymbol{\pi}^{\boldsymbol{\eta}}{}_{\mathrm{cl}}\delta\eta) + \int_{\mathcal{B}} (\underline{\boldsymbol{\Pi}}^{ab}\delta\gamma_{ab} + \underline{\boldsymbol{\Pi}}^{\boldsymbol{\eta}}\delta\eta),
\end{aligned}
\tag{2.65}
$$

and we define the quasilocal stress-energy tensor:

$$
\tau^{ab} \stackrel{\mathrm{def}}{=} \frac{2}{\sqrt{-\gamma}}\frac{\delta\mathfrak{L}_{\mathrm{cl}}}{\delta\gamma_{ab}} = \frac{2}{\sqrt{-\gamma}}(\boldsymbol{\Pi}_{\mathrm{cl}}{}^{ab} - \underline{\boldsymbol{\Pi}}^{ab}) = 2(\Pi_{\mathrm{cl}}{}^{ab} - \underline{\Pi}^{ab}). \tag{2.66}
$$

Again, the normal and tangential projections to $\mathcal{B}$ give the energy, momentum, and stress densities:

$$
\boldsymbol{E} \stackrel{\mathrm{def}}{=} \sqrt{\boldsymbol{\sigma}}\,\tilde{u}_a\tilde{u}_b\tau^{ab} = -\frac{\delta\mathfrak{L}_{\mathrm{cl}}}{\delta N}, \tag{2.67a}
$$

$$
\boldsymbol{J}_A \stackrel{\mathrm{def}}{=} -\sqrt{\boldsymbol{\sigma}}\,\sigma_{Aa}\tilde{u}_b\tau^{ab} = \frac{\delta\mathfrak{L}_{\mathrm{cl}}}{\delta N^A}, \tag{2.67b}
$$

$$
\boldsymbol{s}^{AB} \stackrel{\mathrm{def}}{=} \sqrt{-\gamma}\,\sigma^A{}_a\sigma^B{}_b\tau^{ab} = 2\frac{\delta\mathfrak{L}_{\mathrm{cl}}}{\delta\sigma_{AB}}, \tag{2.67c}
$$

but this time their explicit forms are:

$$
\boldsymbol{E} = 2\kappa\sqrt{\boldsymbol{\sigma}}\,(\eta k - \tilde{u}^\mu\boldsymbol{\nabla}_\mu\eta) - \underline{\boldsymbol{E}}, \tag{2.68a}
$$

$$
\boldsymbol{J}_A = -2\sqrt{\boldsymbol{\sigma}}\,\gamma_{Ai}n_jP^{ij} - \underline{\boldsymbol{J}}_A, \tag{2.68b}
$$

$$
\begin{aligned}
\boldsymbol{s}^{AB} = {} & 2\kappa\sqrt{\boldsymbol{\sigma}}\,\{\eta[k^{AB} + \sigma^{AB}(n^\mu u^\nu\boldsymbol{\nabla}_\nu u_\mu - k)] \\
& + \sigma^{AB}\tilde{u}^\mu\boldsymbol{\nabla}_\mu\eta\} - \underline{\boldsymbol{s}}^{AB}.
\end{aligned}
\tag{2.68c}
$$

Counterterms

Definition with respect to a reference spacetime As in the non-dilatonic case, the quantities $\boldsymbol{E}$ and $\boldsymbol{J}_A$ depend exclusively on the canonical variables only if $\underline{\boldsymbol{E}}$ and $\underline{\boldsymbol{J}}_A$ do. Hence, just like in the non-dilatonic case, we may require the counterterm $\underline{\mathfrak{L}}$ to be a linear functional of the (boundary) lapse and shift:

$$
\underline{\mathfrak{L}} \stackrel{\mathrm{def}}{=} -\int_{\mathcal{B}} (\tilde{N}\underline{\boldsymbol{E}} - \tilde{N}^A\underline{\boldsymbol{J}}_A), \tag{2.69}
$$

and its construction with respect to the reference spacetime follows the same steps as in the non-dilatonic case, but with the additional requirement that the embedding of $\mathcal{B}$ into the reference spacetime should be not only isometric, but 'isodilatonic' as well.

From the variation of *ylio*, which now reads:

$$\delta\underline{\mathfrak{L}} = -\int_{\mathcal{B}}\Bigg[\underline{\boldsymbol{E}}\,\delta\tilde{N} + \underline{\boldsymbol{J}}_A\,\delta\tilde{N}^A + \left(\tilde{N}\frac{\delta\underline{\boldsymbol{E}}}{\delta\sigma_{AB}} + \tilde{N}^A\frac{\delta\underline{\boldsymbol{J}}_A}{\delta\sigma_{AB}}\right)\delta\sigma_{AB} + \left(\tilde{N}\frac{\delta\underline{\boldsymbol{E}}}{\delta\eta} + \tilde{N}^A\frac{\delta\underline{\boldsymbol{J}}_A}{\delta\eta}\right)\delta\eta\Bigg], \tag{2.70}$$

we obtain the expressions for $\underline{\boldsymbol{s}}^{AB}$ and $\underline{\boldsymbol{\Pi}}^{\eta}$:

$$\underline{\boldsymbol{s}}^{AB} \overset{\text{def}}{=} 2\frac{\delta\underline{\mathfrak{L}}}{\delta\sigma_{AB}} = 2\tilde{N}\frac{\delta\underline{\boldsymbol{E}}}{\delta\sigma_{AB}} + \tilde{N}^A\frac{\delta\underline{\boldsymbol{J}}_A}{\delta\sigma_{AB}}, \tag{2.71a}$$

$$\underline{\boldsymbol{\Pi}}^{\eta} \overset{\text{def}}{=} \frac{\delta\underline{\mathfrak{L}}}{\delta\eta} = \tilde{N}\frac{\delta\underline{\boldsymbol{E}}}{\delta\eta} + \tilde{N}^A\frac{\delta\underline{\boldsymbol{J}}_A}{\delta\eta}. \tag{2.71b}$$

Note that both $\underline{\boldsymbol{s}}^{AB}$ and $\underline{\boldsymbol{\Pi}}^{\eta}$ are *not* equivalent to those calculated with respect to the reference spacetime.

This method of constructing a counterterm suffers the problems that we saw in the non-dilatonic case, namely the choice of the reference spacetime and the existence and uniqueness of the embedding; moreover, the additional difficulty of having an isodilatonic embedding now arises. This method is adopted by Lau [32].

Intrinsic definition Balasubramanian and Kraus' method, according to which the counterterm is constructed from the boundary intrinsic metric objects by demanding non-diverging charges, is no more univocal when applied to a dilatonic theory. This happens because a scalar field (the dilaton) is now to be included among the boundary intrinsic metric objects, and there is almost no limit to the number of intrinsic terms that can be constructed from a scalar field, and of the terms which give a *finite* contribution to the charges, in particular. We shall clearly see this fact in the calculations to follow.

Equation of motion for the quasilocal tensor

The equation of motion for the quasilocal tensor needs a more careful analysis in this case, because of the presence of the dilaton.

from the fact that the total and the matter actions $\mathfrak{L}$ and $\mathfrak{L}^{\text{mat}}$ are diffeomorphism-invariant, we have the constraint on $\mathcal{B}$:

$$2\boldsymbol{\Delta}_b{\Pi_{\text{cl}}}^{ab} = {\Pi^{\eta}}_{\text{cl}}\gamma^{ab}\boldsymbol{\Delta}_b\eta - {\gamma^a}_{\mu}n_{\nu}T^{\mu\nu}, \tag{2.72}$$

whence the equation of motion

$$\boldsymbol{\Delta}_b\tau^{ab} = {\Pi^{\eta}}_{\text{cl}}\gamma^{ab}\boldsymbol{\Delta}_b\eta - {\gamma^a}_{\mu}n_{\nu}T^{\mu\nu} \tag{2.73}$$

follows from the definition (2.66) of the quasilocal tensor.

Eq (2.73) is very different from Eq. (2.56): it contains an additional dilatonic term, $\Pi^{\eta}{}_{\rm cl}\gamma^{ab}\boldsymbol{\Delta}_b\eta$, which acts as source together with the semi-projection of the matter stress-energy tensor $\gamma^{a}{}_{\mu}n_{\nu}T^{\mu\nu}$. The presence of the additional dilatonic term has two main consequences, which do not appear in Brown and York's analysis: first, even in the absence of matter there can be non-conserved charges, due to the new source term; second, the subtraction of a counterterm becomes more subtle. We go on analysing these two points in more detail.

Conditions for charge conservation

Contract Eq. (2.73) with a Killing vector field for $\mathcal{B}$, ξ, and integrate between two arbitrary spacelike surfaces in $\mathcal{B}$ so as to obtain:

$$\int_{t'}^{t''}\sqrt{-\gamma}\,\xi^{a}\boldsymbol{\Delta}_b\tau_a{}^{b}=\int_{t'}^{t''}\sqrt{-\gamma}\,(\Pi^{\eta}{}_{\rm cl}\xi^{a}\boldsymbol{\Delta}_a\eta-\xi^{a}\gamma_{a\mu}n_{\nu}T^{\mu\nu}), \tag{2.74}$$

which, by Stokes' Theorem, is equivalent to

$$\mathfrak{Q}_{\mathcal{P}_{t''}}(\xi)-\mathfrak{Q}_{\mathcal{P}_{t'}}(\xi)=-\int_{t'}^{t''}\sqrt{-\gamma}\,(\Pi^{\eta}{}_{\rm cl}\xi^{a}\boldsymbol{\Delta}_a\eta-\xi^{a}\gamma_{a\mu}n_{\nu}T^{\mu\nu}), \tag{2.75}$$

where the charge is defined as

$$\mathfrak{Q}_{\mathcal{P}_t}(\xi)\stackrel{\rm def}{=}\int_{\mathcal{P}_t}\sqrt{\boldsymbol{\sigma}}\,\xi^{a}\tilde{u}^{b}\tau_{ab}. \tag{2.76}$$

One can clearly see that the vanishing of the matter source term, $\xi^{a}\gamma_{a\mu}n_{\nu}T^{\mu\nu}$, does not necessarily imply charge conservation anymore, for now we have also a dilatonic term which can be non-vanishing even when the matter term does vanish. For this reason, one can require the following additional condition for the Killing vector field ξ, like Creighton and Mann [24] do:

$$\mathbf{L}_{\xi}\eta\equiv\xi^{a}\boldsymbol{\Delta}_a\eta=0, \tag{2.77}$$

i.e. one demands that the vector field ξ be a symmetry not only for the metric field, but also for the dilaton field. With this requirement the integrands in Eq. (2.74) become

$$\boldsymbol{\Delta}_b\xi^{a}\tau_a{}^{b}=-\xi^{a}\gamma_{a\mu}n_{\nu}T^{\mu\nu}, \tag{2.78}$$

and upon integration one now obtains:

$$\mathfrak{Q}_{\mathcal{P}_{t''}}(\xi)-\mathfrak{Q}_{\mathcal{P}_{t'}}(\xi)=-\int_{t'}^{t''}\sqrt{-\gamma}\,\xi^{a}\gamma_{a\mu}n_{\nu}T^{\mu\nu}, \tag{2.79}$$

which is analogous to Eq. (2.58).

Creighton and Mann's requirement (2.77) is quite natural because the system is characterized by both the metric and the dilaton fields. Moreover Eq. (2.77) is usually satisfied in the definition of the quasilocal energy, since the latter quantity is usually associated with a timelike Killing vector field and the dilaton is usually independent of time. Nevertheless condition (2.77) reduces the number of symmetries in the system.

This fact is quite general: the presence of a non-constant dilaton breaks the system's symmetries (isometries and asymptotic symmetries), and only a part of them survives; this part usually gives conserved charges just like the mass; the remaining part gives either infinite or finite but *non-conserved* charges. In the second eventuality, we can disregard condition (2.77) if we are interested in having a large number of symmetries rather than charges, since we have no unacceptable physical consequences, like diverging charges. In fact, in studying asymptotic symmetries, we shall not be worried about the strict holding of Eq. (2.77), and Eq. (2.75) will be our equation for charge conservation.

2.3.3 Problems with Brown and York's formalism in the asymptotic limit

In Brown and York's definition for the charge associated to a vector field ξ, it is of fundamental importance the requirement that such a vector field be *a Killing vector field for the boundary $\mathcal{B}$*,

$$\mathbf{L}_{\xi}\gamma_{ab} \equiv \mathbf{\Delta}_{(a}\xi_{b)} = 0, \tag{2.80}$$

which implies

$$\mathbf{\Delta}_a \xi^a = 0, \tag{2.81}$$

in fact this requirement is a condition for the following passages, which lead from Eq. (2.73) to Eq. (2.58):

$$\begin{aligned} \xi^a \mathbf{\Delta}_b \tau^b{}_a &= \mathbf{\Delta}_b(\xi^a \tau^b{}_a) - \tau^b{}_a \mathbf{\Delta}_b \xi^a \\ &= \mathbf{\Delta}_b(\xi^a \tau^b{}_a) - \tfrac{1}{2}\tau^{ab}\mathbf{\Delta}_{(a}\xi_{b)} = \mathbf{\Delta}_b(\xi^a \tau^b{}_a), \end{aligned} \tag{2.82}$$

where in the last passage index symmetry of τ^{ab} has been used, besides Eq. (2.80). Even this other following series of passages, that leads from Eq. (2.73) to Eq. (2.74), makes use of condition (2.81):

$$-\xi^a \mathbf{\Delta}_a(\eta \Pi^\eta{}_{\mathrm{cl}}) = -\mathbf{\Delta}_a(\xi^a \eta \Pi^\eta{}_{\mathrm{cl}}) + \eta \Pi^\eta{}_{\mathrm{cl}} \mathbf{\Delta}_a \xi^a = -\mathbf{\Delta}_a(\xi^a \eta \Pi^\eta{}_{\mathrm{cl}}). \tag{2.83}$$

In the following chapter, we should like to use Brown and York's formalism to calculate the charges associated to the generator of an asymptotic symmetry; but such a generator will not be a Killing vector field for $\mathcal{B}$ in general; it will not even be *tangent* to $\mathcal{B}$ in general. For this reason we shall adapt Brown and York's formulation to our need as follows: we shall take the *projection* of the generator ξ onto $\mathcal{B}$, $\xi^{\parallel a} \overset{\mathrm{def}}{=} \gamma^a{}_\mu \xi^\mu$, and require it to be a Killing vector field for $\mathcal{B}$ *only asymptotically*:

$$\mathbf{L}_{\xi^\parallel}\gamma_{ab} \equiv \mathbf{\Delta}_{(a}\xi^{\parallel}{}_{b)} \xrightarrow{r\to\infty} 0, \tag{2.84}$$

whence

$$\boldsymbol{\Delta}_a \xi^{\|a} \xrightarrow{r\to\infty} 0. \tag{2.85}$$

Note that, for the charge-conservation equations to be valid, we could require the conditions:

$$\tau^{ab} \boldsymbol{\Delta}_{(a} \xi^{\|}{}_{b)} \xrightarrow{r\to\infty} 0 \tag{2.86a}$$

and

$$\eta \Pi^{\eta}{}_{\mathrm{cl}} \boldsymbol{\Delta}_a \xi^{\|a} \xrightarrow{r\to\infty} 0. \tag{2.86b}$$

This manoeuvre, which is necessary if we want to use the quasilocal formalism together with the asymptotic symmetries, signals a deficiency of Brown and York's formalism in dealing with general symmetries. We will discuss about this in detail in the concluding Chapter.

Chapter 3

Asymptotic symmetries in anti-de Sitter space

3.1 Hamiltonian formalism

3.1.1 An example: asymptotic symmetries in Minkowski space

As an introductive example of the analysis of asymptotic symmetries we consider a class of solutions for General Relativity in a quadridimensional spacetime whose ground state is flat space[1]

$$ds^2 = -dt^2 + \delta_{ij}\, dx^i\, dx^j. \tag{3.1}$$

The Lagrangian in this case is:

$$\begin{aligned}\mathfrak{L} = \kappa \int_{\mathcal{M}} \sqrt{-\boldsymbol{g}}\, R_{\mathcal{M}} + 2\kappa \int_{\mathcal{S}'}^{\mathcal{S}''} \sqrt{\boldsymbol{h}}\, K - 2\kappa \int_{\mathcal{B}} \sqrt{-\boldsymbol{\gamma}}\, \Theta \\ + 2\kappa \int_{\mathcal{P}'}^{\mathcal{P}''} \sqrt{\boldsymbol{\sigma}}\, \alpha + \underline{\mathfrak{L}},\end{aligned} \tag{3.2}$$

and the action in canonical form is:

$$\mathfrak{S} = \int_{t'}^{t''} \left[\int_{\mathcal{S}} (\boldsymbol{P}^{\mu\nu} \dot{h}_{\mu\nu} - N\boldsymbol{H} - N^i \boldsymbol{H}_i) - \int_{\mathcal{P}} (\tilde{N}\boldsymbol{E} - \tilde{N}^i \boldsymbol{J}_i) \right], \tag{3.3}$$

so that the Hamiltonian takes the following form:

$$\mathfrak{H} = \int_{\mathcal{S}} (N\boldsymbol{H} + N^i \boldsymbol{H}_i) + \int_{\mathcal{P}} (\tilde{N}\boldsymbol{E} - \tilde{N}^i \boldsymbol{J}_i). \tag{3.4}$$

---

[1] Cartesian coordinates will be used in the present section

Firstly, we consider the bulk Hamiltonian only:

$$\mathfrak{H}^1 = \int_{\mathcal{S}} (N\boldsymbol{H} + N^i \boldsymbol{H}_i), \tag{3.5}$$

and we proceed to find its suitable boundary term by Regge and Teitelboim's method (q.v. Sec. 2.2.1). Therefore we must compute the variation of Eq. (3.5):

$$\begin{aligned}\delta\mathfrak{H}^1 = &\int_{\mathcal{S}} (\boldsymbol{A}^{ij}\delta h_{ij} + B_{ij}\boldsymbol{P}^{ij}) \\ &- \int_{\mathcal{P}} \frac{\sqrt{\boldsymbol{\sigma}}}{\sqrt{\boldsymbol{h}}}\tilde{n}_l [\kappa \boldsymbol{G}^{ijkl}(N\mathbf{D}_k\delta h_{ij} - \partial_k N\,\delta h_{ij}) \\ &+ (2N^i\boldsymbol{P}^{kl} - N^l\boldsymbol{P}^{ik})\delta h_{ik} + 2N_i\,\delta\boldsymbol{P}^{il}],\end{aligned} \tag{3.6}$$

and we must establish the asymptotic conditions for a given class of solutions: we choose the class of static, spherically simmetric, asymptotically flat solutions, which are described by the following asymptotic conditions:

$$g_{tt} = -1 + O\left(\frac{1}{r}\right), \tag{3.7}$$

$$g_{ij} = \delta_{ij} + O\left(\frac{1}{r}\right). \tag{3.8}$$

The asymptotic symmetry group of this asymptotic conditions is generated by

$$A_\mu = \left[1 + O\left(\frac{1}{r}\right)\right]\frac{\partial}{\partial x^\mu}, \tag{3.9a}$$

$$B_\beta = \left[\beta^\mu_\nu x^\nu + O\left(\frac{1}{r}\right)\right]\frac{\partial}{\partial x^\mu} \quad \text{with } \beta_{\mu\nu} = -\beta_{\nu\mu}, \tag{3.9b}$$

where one should notice the presence of a proper gauge part $O(1/r)$; these generators form the Poincaré group.

For simplicity, we consider the action of the generator A_0 only, so that the lapse and shift are, by Eqs. (1.22),

$$N = 1 + O\left(\frac{1}{r}\right), \tag{3.10a}$$

$$N^i = O\left(\frac{1}{r}\right); \tag{3.10b}$$

upon calculation, one can see that the surface term of Eq. (3.6) which does not vanish is:

$$-\kappa \int_{\mathcal{P}} \frac{\sqrt{\boldsymbol{\sigma}}}{\sqrt{\boldsymbol{h}}}\tilde{n}_l \boldsymbol{G}^{ijkl} N\mathbf{D}_k\delta h_{ij}, \tag{3.11}$$

and integrating its variation we find

$$-\mathfrak{J} = -\kappa \int_{\mathcal{P}} \sqrt{\boldsymbol{\sigma}}\, \tilde{n}^k N(\mathbf{D}_i h_{jk} - \mathbf{D}_k h_{ij}) h^{ij}, \tag{3.12}$$

which is just the term to be subtracted from the Hamiltonian (3.5) so as to make the latter well-defined.

If we calculate the charge associated with the generator A_0 for Schwarzschild's solution (which belong to the class of solutions here considered), we find that it is just the black-hole mass:

$$\mathfrak{J}\left[\frac{\partial}{\partial t}\right] = \mathfrak{H}\left[\frac{\partial}{\partial t}\right] = \kappa \int_{\mathcal{P}} \sqrt{\boldsymbol{\sigma}}\, \tilde{n}^k (\mathbf{D}_i h_{jk} - \mathbf{D}_k h_{ij}) h^{ij} = M. \tag{3.13}$$

3.1.2 Asymptotic symmetries in three-dimensional anti-de Sitter space

First choice for the asymptotic conditions

We go on to make an analogous analysis for a three-dimensional theory with a cosmological constant; our analysis follows Brown and Henneaux's [14].

As we saw in Sec. 1.3.2, a simple solution for this case is (three-dimensional) anti-de Sitter space; there are black-hole solutions as well,

$$\begin{aligned} ds^2 &= -(\lambda^2 r^2 + \alpha^2) dt^2 + (\lambda^2 r^2 + \alpha^2 - A^2\lambda^2)^{-1} dr^2 \\ &\quad + 2\alpha A dt\, d\phi + (r^2 - A^2) d\phi^2, \end{aligned} \tag{3.14}$$

and it is quite natural to consider these solutions as excitations of the ground state

$$ds^2 = -\lambda^2 r^2 dt^2 + (\lambda^2 r^2)^{-1} dr^2 + r^2 d\phi^2. \tag{3.15}$$

Hence the excited states share the following asymptotic conditions:

$$g_{tt} = -\lambda^2 r^2 + O(r^0), \tag{3.16a}$$
$$g_{t\phi} = O(r^0), \tag{3.16b}$$
$$g_{tr} = 0, \tag{3.16c}$$
$$g_{\phi\phi} = r^2 + O(r^0), \tag{3.16d}$$
$$g_{\phi r} = 0, \tag{3.16e}$$
$$g_{rr} = \frac{1}{\lambda^2 r^2} + O\left(\frac{1}{r^4}\right). \tag{3.16f}$$

The asymptotic symmetry group for these conditions is generated by the vector fields $\partial/\partial t$ and $\partial/\partial \phi$ (proper gauge part omitted).

Consider the Hamiltonian

$$\mathfrak{H}^1 = \int_{\mathcal{S}} (N\boldsymbol{H} + N^i \boldsymbol{H}_i), \tag{3.17}$$

whose variation is

$$
\begin{aligned}
\delta\mathfrak{H}^1 = {} & \int_{\mathcal{S}} (\boldsymbol{A}^{ij}\delta h_{ij} + B_{ij}\boldsymbol{P}^{ij}) \\
& - \int_{\mathcal{P}} \frac{\sqrt{\boldsymbol{\sigma}}}{\sqrt{\boldsymbol{h}}} \tilde{n}_l [\kappa \boldsymbol{G}^{ijkl}(N\mathbf{D}_k \delta h_{ij} - \partial_k N\, \delta h_{ij}) \\
& + (2N^i \boldsymbol{P}^{kl} - N^l \boldsymbol{P}^{ik})\delta h_{ik} + 2N_i\, \delta\boldsymbol{P}^{il}].
\end{aligned}
\tag{3.18}
$$

If we consider the evolution led by a generator which is a linear combination of $\partial/\partial t$ and $\partial/\partial\phi$, i.e. with lapse and shift give by:

$$N = \lambda r + O\left(\frac{1}{r}\right), \tag{3.19a}$$

$$N^\phi = O\left(\frac{1}{r^2}\right), \tag{3.19b}$$

$$N^r = O\left(\frac{1}{r}\right), \tag{3.19c}$$

we find that the non-vanishing term for the conditions (3.16) is

$$-\int_{\mathcal{P}} \frac{\sqrt{\boldsymbol{\sigma}}}{\sqrt{\boldsymbol{h}}} \tilde{n}_l (\kappa \boldsymbol{G}^{ijkl} N\mathbf{D}_k \delta h_{ij} + 2N_i \delta\boldsymbol{P}^{il}), \tag{3.20}$$

which leads to the surface term for the Hamiltonian:

$$\mathfrak{J} = \int_{\mathcal{P}} \frac{\sqrt{\underline{\boldsymbol{\sigma}}}}{\sqrt{\underline{\boldsymbol{h}}}} \tilde{n}_l [\kappa \boldsymbol{G}^{ijkl} N\mathbf{D}_k h_{ij} + 2N_i(\boldsymbol{P}^{il} - \underline{\boldsymbol{P}}^{il})]. \tag{3.21}$$

Calculation of the charges associated to $\partial/\partial t$ and $\partial/\partial\phi$ yields:

$$\mathfrak{J}\left[\frac{\partial}{\partial t}\right] = \mathfrak{H}\left[\frac{\partial}{\partial t}\right] = \alpha \tag{3.22a}$$

$$\mathfrak{J}\left[\frac{\partial}{\partial \phi}\right] = \mathfrak{H}\left[\frac{\partial}{\partial \phi}\right] = \alpha A \tag{3.22b}$$

and we find the mass and the angular momentum.

Second, more general choice of asymptotic conditions

We should like to consider a more general asymptotic behaviour than (3.16), hoping thus to enrich the system's asymptotic symmetry group with new symmetries: it would be desirable to recover the full isometry group of anti-de Sitter space — just as it happened with the Poincaré group for Minkowski space in the previous example.

We can try to realize this purpose by seeking for the smallest class of asymptotic conditions which contains conditions (3.16) and is closed under the action of anti-de Sitter space's isometries. In order to do this we can simply apply the isometry generators to conditions (3.16) repeatedly; thus we find the new asymptotic conditions:

$$g_{tt} = -\lambda^2 r^2 + O(r^0), \tag{3.23a}$$
$$g_{t\phi} = O(r^0), \tag{3.23b}$$
$$g_{tr} = O\left(\frac{1}{r^3}\right), \tag{3.23c}$$
$$g_{\phi\phi} = r^2 + O(r^0), \tag{3.23d}$$
$$g_{\phi r} = O\left(\frac{1}{r^3}\right), \tag{3.23e}$$
$$g_{rr} = \frac{1}{\lambda^2 r^2} + O\left(\frac{1}{r^4}\right). \tag{3.23f}$$

But, to our surprise, the symmetry group for the conditions (3.23) is not just $SO(2,2)$ (even though it does contain $SO(2,2)$ by construction): rather, it is the infinite-dimensional group of conformal transformations in two dimensions:

$$\begin{aligned}\xi = &\left[\varepsilon(t,\phi) + \frac{1}{\lambda^2 r^2}\bar{\varepsilon}(t,\phi) + O\left(\frac{1}{r^4}\right)\right]\frac{\partial}{\partial t} \\ &+ \left[\omega(t,\phi) + \frac{1}{\lambda^2 r^2}\bar{\omega}(t,\phi) + O\left(\frac{1}{r^4}\right)\right]\frac{\partial}{\partial \phi} \\ &+ \left[r\rho(t,\phi) + O\left(\frac{1}{r}\right)\right]\frac{\partial}{\partial r},\end{aligned} \tag{3.24a}$$

with

$$\lambda^2 \partial_\phi \varepsilon(t,\phi) = \partial_t \omega(t,\phi), \tag{3.24b}$$
$$\partial_t \varepsilon(t,\phi) = \partial_\phi \omega(t,\phi) = -\rho(t,\phi), \tag{3.24c}$$
$$\bar{\varepsilon}(t,\phi) = -\frac{1}{2\lambda^2}\partial_t \rho(t,\phi), \tag{3.24d}$$
$$\bar{\omega}(t,\phi) = \tfrac{1}{2}\partial_\phi \rho(t,\phi). \tag{3.24e}$$

Because of the periodicity in the angular variable, we can expand the generators (3.24) in a Fourier series, obtaining the following countable basis:

$$\begin{aligned}A_n = A_{-n} = &\left[\frac{1}{\lambda}\left(1 - \frac{n^2}{2\lambda^2 r^2}\right)\cos(n\lambda t)\cos(n\phi) + O\left(\frac{1}{r^4}\right)\right]\frac{\partial}{\partial t} \\ &- \left[\left(1 + \frac{n^2}{2\lambda^2 r^2}\right)\sin(n\lambda t)\sin(n\phi) + O\left(\frac{1}{r^4}\right)\right]\frac{\partial}{\partial \phi} \\ &+ \left[rn\sin(n\lambda t)\cos(n\phi) + O\left(\frac{1}{r}\right)\right]\frac{\partial}{\partial r}\end{aligned} \tag{3.25a}$$

$$\begin{aligned}
B_n = B_{-n} = &\left[\frac{1}{\lambda}\left(1 - \frac{n^2}{2\lambda^2 r^2}\right)\sin(n\lambda t)\sin(n\phi) + O\left(\frac{1}{r^4}\right)\right]\frac{\partial}{\partial t} \\
&- \left[\left(1 + \frac{n^2}{2\lambda^2 r^2}\right)\cos(n\lambda t)\cos(n\phi) + O\left(\frac{1}{r^4}\right)\right]\frac{\partial}{\partial \phi} \qquad (3.25b) \\
&+ \left[rn\cos(n\lambda t)\sin(n\phi) + O\left(\frac{1}{r}\right)\right]\frac{\partial}{\partial r} \\
C_n = -C_{-n} = &\left[\frac{1}{\lambda}\left(1 - \frac{n^2}{2\lambda^2 r^2}\right)\sin(n\lambda t)\cos(n\phi) + O\left(\frac{1}{r^4}\right)\right]\frac{\partial}{\partial t} \\
&+ \left[\left(1 + \frac{n^2}{2\lambda^2 r^2}\right)\cos(n\lambda t)\sin(n\phi) + O\left(\frac{1}{r^4}\right)\right]\frac{\partial}{\partial \phi} \qquad (3.25c) \\
&- \left[rn\cos(n\lambda t)\cos(n\phi) + O\left(\frac{1}{r}\right)\right]\frac{\partial}{\partial r} \\
D_n = -D_{-n} = &\left[\frac{1}{\lambda}\left(1 - \frac{n^2}{2\lambda^2 r^2}\right)\cos(n\lambda t)\sin(n\phi) + O\left(\frac{1}{r^4}\right)\right]\frac{\partial}{\partial t} \\
&+ \left[\left(1 + \frac{n^2}{2\lambda^2 r^2}\right)\sin(n\lambda t)\cos(n\phi) + O\left(\frac{1}{r^4}\right)\right]\frac{\partial}{\partial \phi} \qquad (3.25d) \\
&+ \left[rn\sin(n\lambda t)\sin(n\phi) + O\left(\frac{1}{r}\right)\right]\frac{\partial}{\partial r}.
\end{aligned}$$

The basis generators satisfy the following commutation relations:

$$[A_n, A_m] = \tfrac{1}{2}(n-m)C_{n+m} + \tfrac{1}{2}(n+m)C_{n-m}, \qquad (3.26a)$$

$$[B_n, B_m] = \tfrac{1}{2}(n-m)C_{n+m} + \tfrac{1}{2}(n+m)C_{n-m}, \qquad (3.26b)$$

$$[C_n, C_m] = -\tfrac{1}{2}(n-m)C_{n+m} + \tfrac{1}{2}(n+m)C_{n-m}, \qquad (3.26c)$$

$$[D_n, D_m] = -\tfrac{1}{2}(n-m)C_{n+m} + \tfrac{1}{2}(n+m)C_{n-m}, \qquad (3.26d)$$

$$[A_n, B_m] = -\tfrac{1}{2}(n-m)D_{n+m} - \tfrac{1}{2}(n+m)D_{n-m}, \qquad (3.26e)$$

$$[A_n, C_m] = -\tfrac{1}{2}(n-m)A_{n+m} + \tfrac{1}{2}(n+m)A_{n-m}, \qquad (3.26f)$$

$$[A_n, D_m] = \tfrac{1}{2}(n-m)B_{n+m} - \tfrac{1}{2}(n+m)B_{n-m}, \qquad (3.26g)$$

$$[B_n, C_m] = -\tfrac{1}{2}(n-m)B_{n+m} + \tfrac{1}{2}(n+m)B_{n-m}, \qquad (3.26h)$$

$$[B_n, D_m] = \tfrac{1}{2}(n-m)A_{n+m} - \tfrac{1}{2}(n+m)A_{n-m}, \qquad (3.26i)$$

$$[C_n, D_m] = -\tfrac{1}{2}(n-m)D_{n+m} + \tfrac{1}{2}(n+m)D_{n-m}, \qquad (3.26j)$$

and so they form an algebra isomorphic to a direct sum of two Virasoro algebras, which is just the algebra of conformal transformations in two dimensions (apart from the central charge).

The lapse and shift associated to the generators (3.24) are

$$N = \lambda r + O\left(\frac{1}{r}\right), \qquad (3.27a)$$

$$N^\phi = O\left(\frac{1}{r^2}\right), \qquad (3.27b)$$

$$N^r = O\left(\frac{1}{r}\right), \tag{3.27c}$$

so that the non-vanishing term in the variation (3.18) is

$$\int_{\mathcal{P}} \frac{\sqrt{\boldsymbol{\sigma}}}{\sqrt{\boldsymbol{h}}} \tilde{n}_l [\kappa \boldsymbol{G}^{ijkl} (N \mathbf{D}_k \delta h_{ij} - \partial_k N \, \delta h_{ij}) + 2 N_i \delta \boldsymbol{P}^{il}], \tag{3.28}$$

and the well-defined Hamiltonian is

$$\begin{aligned} \mathfrak{H} = & \int_{\mathcal{S}} (N \boldsymbol{H} + N^i \boldsymbol{H}_i) \\ & - \int_{\mathcal{P}} \frac{\sqrt{\boldsymbol{\sigma}}}{\sqrt{\boldsymbol{h}}} \tilde{n}_l \{ \kappa \boldsymbol{G}^{ijkl} [N \mathbf{D}_k h_{ij} - \partial_k N \, (h_{ij} - \underline{h}_{ij})] \\ & + 2 N_i (\boldsymbol{P}^{il} - \underline{\boldsymbol{P}}^{il}) \}. \end{aligned} \tag{3.29}$$

The only non-vanishing charges for a solution like (3.14) are the ones associated to the generators $A_0 = \frac{1}{\lambda} \frac{\partial}{\partial t}$ and $B_0 = \frac{\partial}{\partial \phi}$:

$$\mathfrak{J}[A_0] = \frac{\alpha}{\lambda}, \tag{3.30a}$$

$$\mathfrak{J}[B_0] = \alpha A; \tag{3.30b}$$

so we find again the mass and the angular momentum. But we can look for new charges coming from solutions different from (3.14) but which belong to the asymptotic class (3.23). We can generate these kinds of solutions by infinitesimally deforming the ground state by means of a generator (3.25):

$$g_{\mu\nu} = \underline{g}_{\mu\nu} + \mathbf{L}_{\epsilon\xi} \underline{g}_{\mu\nu}; \tag{3.31}$$

thus we find the central charges

$$\mathfrak{J}[A_n] = \frac{1}{\lambda} n^3 \delta_{|n|\,|m|}, \qquad \text{per } g_{\mu\nu} = \underline{g}_{\mu\nu} + \mathbf{L}_{\epsilon C_m} \underline{g}_{\mu\nu}, \tag{3.32a}$$

$$\mathfrak{J}[B_n] = -\frac{1}{\lambda} n^3 \delta_{|n|\,|m|}, \qquad \text{per } g_{\mu\nu} = \underline{g}_{\mu\nu} + \mathbf{L}_{\epsilon D_m} \underline{g}_{\mu\nu}. \tag{3.32b}$$

3.1.3 Asymptotic symmetries in two-dimensional anti-de Sitter space with dilaton field

In the first chapter we mentioned the fact that a classical (i.e. non-dilatonic) gravity theory cannot exist in two dimensions, since the scalar curvature is a topological invariant and The Hilbert-Einstein action has no dynamics. It is necessary to introduce at least one more degree of freedom into the theory, e.g. a dilaton field. As a consequence, the asymptotic analysis of such a two-dimensional theory will have to consider the dilaton's asymptotic behaviour as well. In this context finds its place the work by Cadoni and Mignemi [19], which we are going to follow in this section.

We take as ground state the following solution of the equations of motion:

$$ds^2 = -\lambda^2 r^2 dt^2 + (\lambda^2 r^2)^{-1} dr^2, \tag{3.33a}$$
$$\eta = \eta_0 \lambda r, \tag{3.33b}$$

and as 'excited' solutions:

$$ds^2 = -(\lambda^2 r^2 + \alpha^2) dt^2 + (\lambda^2 r^2 + \alpha^2)^{-1} dr^2, \tag{3.34a}$$
$$\eta = \eta_0 \lambda r, \tag{3.34b}$$

which behave at infinity as follows:

$$g_{tt} = -\lambda^2 r^2 + O(r^0), \tag{3.35a}$$
$$g_{tr} = 0, \tag{3.35b}$$
$$g_{rr} = \frac{1}{\lambda^2 r^2} + O\left(\frac{1}{r^4}\right), \tag{3.35c}$$
$$\eta = \eta_0 \lambda r. \tag{3.35d}$$

The asymptotic conditions above lead to a rather poor symmetry group, just as it happened with conditions (3.16) in the three-dimensional case. If one wants to recover the isometry group $SO(2,1)$ at least, one must assume the following 'larger' conditions:

$$g_{tt} = -\lambda^2 r^2 + O(r^0), \tag{3.36a}$$
$$g_{tr} = O\left(\frac{1}{r^3}\right), \tag{3.36b}$$
$$g_{rr} = \frac{1}{\lambda^2 r^2} + O\left(\frac{1}{r^4}\right), \tag{3.36c}$$
$$\eta = O(r). \tag{3.36d}$$

The asymptotic symmetry group for conditions (3.36) is generated by:

$$\xi = \left[\varepsilon(t) + \frac{1}{2\lambda^4 r^2}\frac{d^2\varepsilon(t)}{dt^2} + O\left(\frac{1}{r^4}\right)\right]\frac{\partial}{\partial t} - \left[r\frac{d\varepsilon(t)}{dt} + O\left(\frac{1}{r}\right)\right]\frac{\partial}{\partial r}. \tag{3.37}$$

Again, to our surprise, the group is infinite-dimensional and coincides with the conformal group in one dimension; it contains $SO(2,1)$ as subgroup. Fourier analysis of its generators leads to the following countable basis:

$$\begin{aligned} A_n = A_{-n} = {} & \left[\frac{1}{\lambda}\left(1 - \frac{n^2}{2\lambda^2 r^2}\right)\cos(n\lambda t) + O\left(\frac{1}{r^4}\right)\right]\frac{\partial}{\partial t} \\ & + \left[rn\sin(n\lambda t) + O\left(\frac{1}{r}\right)\right]\frac{\partial}{\partial r}, \end{aligned} \tag{3.38a}$$

$$\begin{aligned} B_n = -B_{-n} = {} & \left[\frac{1}{\lambda}(1 - \frac{n^2}{2\lambda^2 r^2}\right)\sin(n\lambda t) + O\left(\frac{1}{r^4}\right)\right]\frac{\partial}{\partial t} \\ & - \left[rn\cos(n\lambda t) + O\left(\frac{1}{r}\right)\right]\frac{\partial}{\partial r}; \end{aligned} \tag{3.38b}$$

the commutation relations are:

$$[A_n, A_m] = \tfrac{1}{2}(n-m)B_{n+m} + \tfrac{1}{2}(n+m)B_{n-m}, \tag{3.39a}$$
$$[B_n, B_m] = -\tfrac{1}{2}(n-m)B_{n+m} + \tfrac{1}{2}(n+m)B_{n-m}, \tag{3.39b}$$
$$[A_n, B_m] = -\tfrac{1}{2}(n-m)A_{n+m} + \tfrac{1}{2}(n+m)A_{n-m}, \tag{3.39c}$$

so that the generators form a Virasoro algebra, corresponding to the algebra of the conformal group in one dimension (apart from the central charge).

Go on to consider, as usual, the variation of the bulk Hamiltonian:

$$\begin{aligned}
\delta\mathfrak{H}^1 = &\int_{\mathcal{S}} (\boldsymbol{A}^{ij}\delta h_{ij} + B_{ij}\boldsymbol{P}^{ij}) \\
&- \int_{\mathcal{P}} \frac{\sqrt{\boldsymbol{\sigma}}}{\sqrt{\boldsymbol{h}}}\tilde{n}_l \{\kappa \boldsymbol{G}^{ijkl}[N\eta \mathbf{D}_k h_{ij} - \partial_k(N\eta)\,\delta h_{ij}] \\
&+ \kappa\sqrt{\boldsymbol{h}}\,(2h^{il}h^{jk} - h^{ij}h^{kl})\delta h_{ij}\,N\,\partial_k\eta \\
&+ 2\kappa\sqrt{\boldsymbol{h}}\,(\partial_j N\,\delta\eta - N\partial_j\,\delta\eta) \\
&+ (2N^i\boldsymbol{P}^{kl} - N^l\boldsymbol{P}^{ik})\delta h_{ik} \\
&+ 2N_i\,\delta\boldsymbol{P}^{il} - N^l\boldsymbol{P}^{\boldsymbol{\eta}}\,\delta\eta\};
\end{aligned} \tag{3.40}$$

if the canonical evolution is generated by a vector field like (3.37), the lapse and shift turn out to be:

$$N = \lambda r + O\left(\frac{1}{r}\right), \tag{3.41a}$$
$$N^r = O\left(\frac{1}{r}\right), \tag{3.41b}$$

and the asymptotically non-vanishing variational term is:

$$\begin{aligned}
-\int_{\mathcal{P}} \frac{\sqrt{\boldsymbol{\sigma}}}{\sqrt{\boldsymbol{h}}}\tilde{n}_l [&\kappa\sqrt{\boldsymbol{h}}\,(2h^{il}h^{jk} - h^{ij}h^{kl})\delta h_{ij}\,N\,\partial_k\eta \\
&+ 2\kappa\sqrt{\boldsymbol{h}}\,(\partial_j N\,\delta\eta - N\partial_j\,\delta\eta) + 2N_i\,\delta\boldsymbol{P}^{il}];
\end{aligned} \tag{3.42}$$

a problem arises at this point: the variation of the dilaton does not vanish asymptotically, and in principle it is not possible to integrate the surface term above; only by requiring the dilaton to have the form

$$\eta = [1 + \delta\rho(t)]O(r), \tag{3.43}$$

i.e. to be 'infinitesimally near' to unity, can we integrate the variational term above and find the term to be added to the Hamiltonian:

$$\begin{aligned}
\mathfrak{J} = \int_{\mathcal{P}} \frac{\sqrt{\boldsymbol{\sigma}}}{\sqrt{\boldsymbol{h}}}\tilde{n}_l \{&\kappa\sqrt{\boldsymbol{h}}\,(2h^{il}h^{jk} - h^{ij}h^{kl})(h_{ij} - \underline{h}_{ij})N\,\partial_k\eta \\
&+ 2\kappa\sqrt{\boldsymbol{h}}\,[\partial_j N(\eta - \underline{\eta})N\partial_j(\eta - \underline{\eta})] \\
&+ 2N_i(\boldsymbol{P}^{il} - \underline{\boldsymbol{P}}^{il})\}.
\end{aligned} \tag{3.44}$$

Calculation of the charges associated to the generators (3.38) for a solution like (3.34) gives:

$$\mathfrak{J}[A_n] = \mathfrak{H}[A_n] = \frac{1}{2}\eta_0 \alpha^2 \cos(n\lambda t), \tag{3.45a}$$

$$\mathfrak{J}[B_n] = \mathfrak{H}[B_n] = \frac{1}{2}\eta_0 \alpha^2 \sin(n\lambda t); \tag{3.45b}$$

the only non-trivial conserved charge is the one associated to $A_0 = \frac{1}{\lambda}\frac{\partial}{\partial t}$, corresponding to the mass; besides we obtain an infinite number of non-conserved charges. Their presence is related to the integration problems concerning the variational surface term, which we saw above; or it can be related to the fact that Hamiltonian evolution and Lie transport do not coincide for the generators of the asymptotic symmetries (which was true for Brown and Henneaux's case instead). Cadoni and Mignemi [20] analyse this problem and as a solution they redefine the charge associated to the generator ξ as:

$$\mathfrak{J}'[\xi] \stackrel{\text{def}}{=} \frac{\lambda}{2\pi} \int_t^{t+\frac{2\pi}{\lambda}} \mathfrak{J}[\xi], \tag{3.46}$$

thus introducing a sort of mean value. By means of this new definition we obtain the conserved charges:

$$\mathfrak{J}'[A_0] = \frac{M}{\lambda}, \tag{3.47a}$$

$$\mathfrak{J}'[A_m] = 0 \quad \text{with } m \neq 0, \tag{3.47b}$$

$$\mathfrak{J}'[B_n] = 0. \tag{3.47c}$$

Now we deform the ground state infinitesimally by means of the generator $\epsilon\xi^\mu$:

$$g_{\mu\nu} = \underline{g}_{\mu\nu} + \mathbf{L}_{\epsilon\xi}\underline{g}_{\mu\nu}; \tag{3.48}$$

and we find find the non-trivial charges associated to this deformation:

$$\mathfrak{J}'[A_n] = \eta_0 n^3 \delta_{|n|\,|m|} \quad \text{for } g_{\mu\nu} = \underline{g}_{\mu\nu} + \mathbf{L}_{\epsilon B_m}\underline{g}_{\mu\nu}. \tag{3.49}$$

3.1.4 Asymptotic symmetries in three-dimensional anti-de Sitter space with dilaton field

The analysis of the asymptotic symmetries of a dilaton gravity theory in three-dimensional anti-de Sitter space allows us to make the simplest untrivial comparison with the respective non-dilatonic theory. The comparison is not trivial because of the space considered, which is not just flat space, and is the simplest because three is the lowest number of dimensions in which such a comparison can me made (in two dimensions there can be no non-dilatonic gravity theories).

The (metric) ground state we consider is the same as the one of non-dilatonic theory:

$$ds^2 = -\lambda^2 r^2 dt^2 + (\lambda^2 r^2 + 1)^{-1} dr^2 + r^2 d\phi^2, \tag{3.50a}$$

$$\eta = \lambda r; \tag{3.50b}$$

and the black-hole solutions are:

$$ds^2 = -\left(\lambda^2 r^2 - \frac{\alpha^2}{r}\right) dt^2 + \left(\lambda^2 r^2 - \frac{\alpha^2}{r}\right)^{-1} dr^2 + r^2 d\phi^2, \tag{3.51a}$$

$$\eta = \lambda r. \tag{3.51b}$$

If we look for the strictest class of asymptotic conditions which contains solutions like (3.51) and is invariant under the action of the ground state's isometry group $SO(2,2)$, we find:

$$g_{tt} = -\lambda^2 r^2 + O\left(\frac{1}{r}\right), \tag{3.52a}$$

$$g_{t\phi} = O\left(\frac{1}{r}\right), \tag{3.52b}$$

$$g_{tr} = O\left(\frac{1}{r^4}\right), \tag{3.52c}$$

$$g_{\phi\phi} = r^2 + O\left(\frac{1}{r}\right), \tag{3.52d}$$

$$g_{\phi r} = O\left(\frac{1}{r^4}\right), \tag{3.52e}$$

$$g_{rr} = \frac{1}{\lambda^2 r^2} + O\left(\frac{1}{r^5}\right), \tag{3.52f}$$

and, for the dilaton:

$$\eta = O(r). \tag{3.52g}$$

We immediately see that these conditions are completely different from the (3.23): their gauge parts fall off faster by one power of $1/r$. The asymptotic symmetries turn out to be different as well:

$$A_1 = \left[\frac{1}{\lambda} + O\left(\frac{1}{r^4}\right)\right]\frac{\partial}{\partial t} + \left[O\left(\frac{1}{r^4}\right)\right]\frac{\partial}{\partial \phi} + \left[O\left(\frac{1}{r}\right)\right]\frac{\partial}{\partial r}, \tag{3.53a}$$

$$A_2 = \left[O\left(\frac{1}{r^4}\right)\right]\frac{\partial}{\partial t} + \left[1 + O\left(\frac{1}{r^4}\right)\right]\frac{\partial}{\partial \phi} + \left[O\left(\frac{1}{r}\right)\right]\frac{\partial}{\partial r}, \tag{3.53b}$$

$$A_3 = \left[\frac{\phi}{\lambda} + O\left(\frac{1}{r^4}\right)\right]\frac{\partial}{\partial t} + \left[\lambda t + O\left(\frac{1}{r^4}\right)\right]\frac{\partial}{\partial \phi} + \left[O\left(\frac{1}{r}\right)\right]\frac{\partial}{\partial r}, \tag{3.53c}$$

$$A_4 = \left[t + O\left(\frac{1}{r^4}\right)\right]\frac{\partial}{\partial t} + \left[\phi + O\left(\frac{1}{r^4}\right)\right]\frac{\partial}{\partial \phi} - \left[r + O\left(\frac{1}{r}\right)\right]\frac{\partial}{\partial r}, \tag{3.53d}$$

$$
\begin{aligned}
A_5 = {} & \left[\lambda t^2 + \frac{\phi^2}{\lambda} + \frac{1}{\lambda^3 r^2} + O\left(\frac{1}{r^4}\right)\right]\frac{\partial}{\partial t} + \left[2\lambda t\phi + O\left(\frac{1}{r^4}\right)\right]\frac{\partial}{\partial \phi} \\
& - \left[2\lambda t r + O\left(\frac{1}{r}\right)\right]\frac{\partial}{\partial r},
\end{aligned} \tag{3.53e}
$$

$$
\begin{aligned}
A_6 = {} & \left[2t\phi + O\left(\frac{1}{r^4}\right)\right]\frac{\partial}{\partial t} + \left[\lambda^2 t^2 + \phi^2 - \frac{1}{\lambda^2 r^2} + O\left(\frac{1}{r^4}\right)\right]\frac{\partial}{\partial \phi} \\
& - \left[2\phi r + O\left(\frac{1}{r}\right)\right]\frac{\partial}{\partial r};
\end{aligned} \tag{3.53f}
$$

these are exactly the generators of the group $SO(2,2)$; their algebra is indeed given by the commutation rules:

$$[A_1, A_2] = 0, \tag{3.54a}$$
$$[A_1, A_3] = A_2, \tag{3.54b}$$
$$[A_1, A_4] = A_1, \tag{3.54c}$$
$$[A_1, A_5] = 2A_4, \tag{3.54d}$$
$$[A_1, A_6] = 2A_3, \tag{3.54e}$$
$$[A_2, A_3] = A_1, \tag{3.54f}$$
$$[A_2, A_4] = A_2, \tag{3.54g}$$
$$[A_2, A_5] = 2A_3, \tag{3.54h}$$
$$[A_2, A_6] = 2A_4, \tag{3.54i}$$
$$[A_3, A_4] = 0, \tag{3.54j}$$
$$[A_3, A_5] = A_6, \tag{3.54k}$$
$$[A_3, A_6] = A_5, \tag{3.54l}$$
$$[A_4, A_5] = A_5, \tag{3.54m}$$
$$[A_4, A_6] = A_6, \tag{3.54n}$$
$$[A_5, A_6] = 0. \tag{3.54o}$$

Now let us consider the Hamiltonian; its variation is:

$$
\begin{aligned}
\delta\mathfrak{H}^1 = {} & \int_{\mathcal{S}} (\boldsymbol{A}^{ij}\delta h_{ij} + B_{ij}\boldsymbol{P}^{ij}) \\
& - \int_{\mathcal{P}} \frac{\sqrt{\boldsymbol{\sigma}}}{\sqrt{\boldsymbol{h}}}\tilde{n}_l \{\kappa \boldsymbol{G}^{ijkl}[N\eta \mathbf{D}_k h_{ij} - \partial_k(N\eta)\,\delta h_{ij}] \\
& + \kappa\sqrt{\boldsymbol{h}}\,(2h^{il}h^{jk} - h^{ij}h^{kl})\delta h_{ij}\, N\,\partial_k\eta \\
& + 2\kappa\sqrt{\boldsymbol{h}}\,(\partial_j N\,\delta\eta - N\partial_j\,\delta\eta) \\
& + (2N^i\boldsymbol{P}^{kl} - N^l\boldsymbol{P}^{ik})\delta h_{ik} \\
& + 2N_i\,\delta\boldsymbol{P}^{il} - N^l\boldsymbol{P}^{\boldsymbol{\eta}}\,\delta\eta\};
\end{aligned} \tag{3.55}
$$

just as it happened in the two-dimensional case, we can integrate the variation only if the dilaton has the following form:

$$\eta = [1 + \delta\rho(t)]O(r); \tag{3.56}$$

with this condition, the Hamiltonian additional surface term is:

$$\begin{aligned}\mathfrak{J} = \int_{\mathcal{P}} \frac{\sqrt{\boldsymbol{\sigma}}}{\sqrt{\boldsymbol{h}}} \tilde{n}_l \{ & \kappa \boldsymbol{G}^{ijkl} [N\eta \mathbf{D}_k h_{ij} - \partial_k (N\eta)(h_{ij} - \underline{h}_{ij})] \\ & + \kappa \sqrt{\boldsymbol{h}} (2h^{il} h^{jk} - h^{ij} h^{kl})(h_{ij} - \underline{h}_{ij}) N \, \partial_k \eta \\ & + 2\kappa \sqrt{\boldsymbol{h}} [\partial_j N(\eta - \underline{\eta}) - N \partial_j (\eta - \underline{\eta})] \\ & + 2N_i (\boldsymbol{P}^{il} - \underline{\boldsymbol{P}}^{il}) \}.\end{aligned} \tag{3.57}$$

.

For a solution like (3.51) the only non-vanishing conserved charge is the mass, associated to the generator $\partial/\partial t = \lambda A_1$:

$$\mathfrak{J}\left[\frac{\partial}{\partial t}\right] = \mathfrak{H}\left[\frac{\partial}{\partial t}\right] = 2\lambda\alpha^2 = M. \tag{3.58}$$

The result is not as surprising as in the non-dilatonic case: the system's asymptotic boundary is just invariant under $SO(2,2)$, but not under the conformal group; this is due to the asymptotic conditions (3.52), which are too strict.

One could think that assuming the same conditions (3.23) of the non-dilatonic case instead of the (3.52) might again result in a larger asymptotic symmetry group. This idea proves to be wrong as soon as the charges associated to the *present* generators (3.53) for a deformed-ground-state solution are calculated: they *diverge*; the system thus has infinite charges. This means that the present asymptotic conditions (3.52) contain physically unacceptable solutions, a fact that will be shown by explicit calculation through Brown and York's formalism. Hence, if we extended the present asymptotic conditions to the (3.23) the situation would be only worse. Instead, we must modify condition (3.52g) for the dilaton, so that the new asymptotic conditions are:

$$g_{tt} = -\lambda^2 r^2 + O\left(\frac{1}{r}\right), \tag{3.59a}$$

$$g_{t\phi} = O\left(\frac{1}{r}\right), \tag{3.59b}$$

$$g_{tr} = O\left(\frac{1}{r^4}\right), \tag{3.59c}$$

$$g_{\phi\phi} = r^2 + O\left(\frac{1}{r}\right), \tag{3.59d}$$

$$g_{\phi r} = O\left(\frac{1}{r^4}\right), \tag{3.59e}$$

$$g_{rr} = \frac{1}{\lambda^2 r^2} + O\left(\frac{1}{r^5}\right), \tag{3.59f}$$

$$\eta = \lambda r + O\left(\frac{1}{r}\right); \tag{3.59g}$$

as a consequence of this restriction, the asymptotic symmetry group shrinks to the subgroup generated by:

$$A_1 = \left[\frac{1}{\lambda} + O\left(\frac{1}{r^4}\right)\right]\frac{\partial}{\partial t} + \left[O\left(\frac{1}{r^4}\right)\right]\frac{\partial}{\partial \phi} + \left[O\left(\frac{1}{r}\right)\right]\frac{\partial}{\partial r}, \tag{3.60a}$$

$$A_2 = \left[O\left(\frac{1}{r^4}\right)\right]\frac{\partial}{\partial t} + \left[1 + O\left(\frac{1}{r^4}\right)\right]\frac{\partial}{\partial \phi} + \left[O\left(\frac{1}{r}\right)\right]\frac{\partial}{\partial r}, \tag{3.60b}$$

$$A_3 = \left[\frac{\phi}{\lambda} + O\left(\frac{1}{r^4}\right)\right]\frac{\partial}{\partial t} + \left[\lambda t + O\left(\frac{1}{r^4}\right)\right]\frac{\partial}{\partial \phi} + \left[O\left(\frac{1}{r}\right)\right]\frac{\partial}{\partial r}. \tag{3.60c}$$

3.2 Quasilocal formalism

We will now apply Brown and York's quasilocal formalism to the calculation of the charges associated to the asymptotic symmetries in anti-de Sitter space for the three theories previously analysed, namely the three-dimensional dilatonic and non-dilatonic theories, and the two-dimensional dilatonic one.

3.2.1 The three-dimensional non-dilatonic case

Counterterm choice

Before proceeding to apply the formulas for the quasilocal charges, Eqs. (2.60) and (2.61), we must choose an explicit form for the counterterm $\underline{\mathfrak{L}}$, which is needed for the renormalisation of the charges. In Sec. 2.3.1 we outlined the main guide-lines for such a choice; we decide to use Balasubramanian and Kraus' method in this case, i.e. to construct a counterterm from the intrinsic metric objects of the boundary. We make such a choice because the intrinsic method is computationally easier and it should be asymptotically equivalent to the reference-spacetime one (q.v. Lau [33]).

The form of the counterterm for three-dimensional anti-de Sitter space has been univocally determined by Balasubramanian and Kraus [5] by the requirement that the quasilocal tensor τ^{ab} should yield non-diverging charges:

$$\underline{\mathfrak{L}} \stackrel{\text{def}}{=} 2\kappa \int_{\mathcal{B}} \sqrt{-\gamma}\,\lambda. \tag{3.61}$$

Its variation is:

$$\delta\underline{\mathfrak{L}} = \kappa \int_{\mathcal{B}} \sqrt{-\gamma}\,\lambda\gamma^{ab}\,\delta\gamma_{ab}, \tag{3.62}$$

whence we obtain the expression for $\underline{\Pi}^{ab}$:

$$\underline{\Pi}^{ab} \stackrel{\text{def}}{=} \frac{\delta\underline{\mathfrak{L}}}{\delta\gamma_{ab}} = \kappa\lambda\gamma^{ab}; \tag{3.63}$$

thus the quasilocal stress-energy tensor is:

$$\begin{aligned}\tau^{ab} &\stackrel{\text{def}}{=} \frac{2}{\sqrt{-\gamma}}\frac{\delta\mathfrak{L}_{\text{cl}}}{\delta\gamma_{ab}} = \frac{2}{\sqrt{-\gamma}}(\Pi^{ab}_{\text{cl}} - \underline{\Pi}^{ab}) \\ &= 2\kappa(\Theta^{ab} - \Theta\gamma^{ab} - \lambda\gamma^{ab}).\end{aligned} \tag{3.64}$$

Charges

We have already analysed the suitable asymptotic conditions, symmetries, and generators for three-dimensional anti-de Sitter space in Sec. 3.1.2. The metric induced at infinity is:

$$g_{tt} = -\lambda^2 r^2 + O(r^0), \tag{3.65a}$$

$$g_{t\phi} = O(r^0), \tag{3.65b}$$

$$g_{tr} = O\left(\frac{1}{r^3}\right), \tag{3.65c}$$

$$g_{\phi\phi} = r^2 + O(r^0), \tag{3.65d}$$

$$g_{\phi r} = O\left(\frac{1}{r^3}\right), \tag{3.65e}$$

$$g_{rr} = \frac{1}{\lambda^2 r^2} + O\left(\frac{1}{r^4}\right), \tag{3.65f}$$

and it contains solutions like:

$$\begin{aligned}ds^2 &= -(\lambda^2 r^2 + \alpha^2)dt^2 + (\lambda^2 r^2 + \alpha^2 - A^2\lambda^2)^{-1}dr^2 \\ &\quad + 2\alpha A dt\, d\phi + (r^2 - A^2)d\phi^2,\end{aligned} \tag{3.66}$$

which are considered as excitations of the following ground state:

$$ds^2 = -\lambda^2 r^2 dt^2 + (\lambda^2 r^2)^{-1}dr^2 + r^2 d\phi^2. \tag{3.67}$$

The asymptotic symmetry group turns out to be the conformal group in two dimensions, which contains the ground-state isometry subgroup $SO(2,2)$, and whose generators are:

$$\begin{aligned}\xi = &\left[\varepsilon(t,\phi) + \frac{1}{\lambda^2 r^2}\bar{\varepsilon}(t,\phi) + O\left(\frac{1}{r^4}\right)\right]\frac{\partial}{\partial t} \\ &+ \left[\omega(t,\phi) + \frac{1}{\lambda^2 r^2}\bar{\omega}(t,\phi) + O\left(\frac{1}{r^4}\right)\right]\frac{\partial}{\partial \phi} \\ &+ \left[r\rho(t,\phi) + O\left(\frac{1}{r}\right)\right]\frac{\partial}{\partial r},\end{aligned} \tag{3.68a}$$

with

$$
\begin{aligned}
\lambda^2 \partial_\phi \varepsilon(t,\phi) &= \partial_t \omega(t,\phi), && (3.68\text{b}) \\
\partial_t \varepsilon(t,\phi) &= \partial_\phi \omega(t,\phi) = -\rho(t,\phi), && (3.68\text{c}) \\
\bar{\varepsilon}(t,\phi) &= -\frac{1}{2\lambda^2}\partial_t \rho(t,\phi), && (3.68\text{d}) \\
\bar{\omega}(t,\phi) &= \tfrac{1}{2}\partial_\phi \rho(t,\phi). && (3.68\text{e})
\end{aligned}
$$

These generators can be Fourier analysed, and we have the countable basis:

$$
\begin{aligned}
A_n = A_{-n} = & \left[\frac{1}{\lambda}\left(1 - \frac{n^2}{2\lambda^2 r^2}\right)\cos(n\lambda t)\cos(n\phi) + O\left(\frac{1}{r^4}\right)\right]\frac{\partial}{\partial t} \\
& - \left[\left(1 + \frac{n^2}{2\lambda^2 r^2}\right)\sin(n\lambda t)\sin(n\phi) + O\left(\frac{1}{r^4}\right)\right]\frac{\partial}{\partial \phi} \qquad (3.69\text{a}) \\
& + \left[rn\sin(n\lambda t)\cos(n\phi) + O\left(\frac{1}{r}\right)\right]\frac{\partial}{\partial r} \\
B_n = B_{-n} = & \left[\frac{1}{\lambda}\left(1 - \frac{n^2}{2\lambda^2 r^2}\right)\sin(n\lambda t)\sin(n\phi) + O\left(\frac{1}{r^4}\right)\right]\frac{\partial}{\partial t} \\
& - \left[\left(1 + \frac{n^2}{2\lambda^2 r^2}\right)\cos(n\lambda t)\cos(n\phi) + O\left(\frac{1}{r^4}\right)\right]\frac{\partial}{\partial \phi} \qquad (3.69\text{b}) \\
& + \left[rn\cos(n\lambda t)\sin(n\phi) + O\left(\frac{1}{r}\right)\right]\frac{\partial}{\partial r} \\
C_n = -C_{-n} = & \left[\frac{1}{\lambda}\left(1 - \frac{n^2}{2\lambda^2 r^2}\right)\sin(n\lambda t)\cos(n\phi) + O\left(\frac{1}{r^4}\right)\right]\frac{\partial}{\partial t} \\
& + \left[\left(1 + \frac{n^2}{2\lambda^2 r^2}\right)\cos(n\lambda t)\sin(n\phi) + O\left(\frac{1}{r^4}\right)\right]\frac{\partial}{\partial \phi} \qquad (3.69\text{c}) \\
& - \left[rn\cos(n\lambda t)\cos(n\phi) + O\left(\frac{1}{r}\right)\right]\frac{\partial}{\partial r} \\
D_n = -D_{-n} = & \left[\frac{1}{\lambda}\left(1 - \frac{n^2}{2\lambda^2 r^2}\right)\cos(n\lambda t)\sin(n\phi) + O\left(\frac{1}{r^4}\right)\right]\frac{\partial}{\partial t} \\
& + \left[\left(1 + \frac{n^2}{2\lambda^2 r^2}\right)\sin(n\lambda t)\cos(n\phi) + O\left(\frac{1}{r^4}\right)\right]\frac{\partial}{\partial \phi} \qquad (3.69\text{d}) \\
& + \left[rn\sin(n\lambda t)\sin(n\phi) + O\left(\frac{1}{r}\right)\right]\frac{\partial}{\partial r}.
\end{aligned}
$$

Since these generators satisfy the commutation relations (3.26) (p. 56), they span a direct sum of two copies of a Virasoro algebra.

According to what we said in Sec. 2.3.3, if the quasilocal is to be used properly, then the generators ξ must satisfy the condition

$$
\mathbf{L}_{\xi^{\parallel}}\gamma_{ab} \equiv \mathbf{\Delta}_{(a}\xi^{\parallel}{}_{b)} \xrightarrow{r\to\infty} 0, \qquad (3.70\text{a})
$$

i.e.

$$\mathbf{\Delta}_a \xi^{\|a} \xrightarrow{r\to\infty} 0, \tag{3.70b}$$

or the condition

$$\tau^{ab}\mathbf{\Delta}_{(a}\xi^{\|}{}_{b)} \xrightarrow{r\to\infty} 0. \tag{3.71}$$

Indeed, for a generator like (3.68) we have:

$$\tau^{ab}\mathbf{\Delta}_{(a}\xi^{\|}{}_{b)} = O\left(\frac{1}{r^4}\right), \tag{3.72}$$

which is satisfactory.

Explicit calculation of the charges for a solution like (3.66) gives

$$\mathfrak{Q}(A_0) \stackrel{\text{def}}{=} \int_{\mathcal{P}} \sqrt{\boldsymbol{\sigma}}\, A_0{}^{\|a} \tilde{u}_b \tau^b{}_a = \alpha, \tag{3.73a}$$

$$\mathfrak{Q}(B_0) \stackrel{\text{def}}{=} \int_{\mathcal{P}} \sqrt{\boldsymbol{\sigma}}\, B_0{}^{\|a} \tilde{u}_b \tau^b{}_a = \alpha A \tag{3.73b}$$

as the only non-vanishing charges, i.e. we find mass and angular momentum just like we expected. If we calculate the charges for an infinitesimal deformation of the ground state we find the untrivial results:

$$\mathfrak{Q}(A_n) = \frac{1}{\lambda} n^3 \delta_{|n|\,|m|}, \qquad \text{per } g_{\mu\nu} = \underline{g}_{\mu\nu} + \mathbf{L}_{\epsilon C_m} \underline{g}_{\mu\nu}, \tag{3.74a}$$

$$\mathfrak{Q}(B_n) = -\frac{1}{\lambda} n^3 \delta_{|n|\,|m|}, \qquad \text{per } g_{\mu\nu} = \underline{g}_{\mu\nu} + \mathbf{L}_{\epsilon D_m} \underline{g}_{\mu\nu}, \tag{3.74b}$$

i.e. the central charges found in Sec. 3.1.2 by Regge and Teitelboim's procedure.

3.2.2 The three-dimensional dilatonic case

Three-dimensional anti-de Sitter space with a dilaton field is the first arena wherein we face the problems discussed in Sec. 2.3.2.

Counterterm

Even in this case we decide to construct the counterterm $\underline{\mathfrak{L}}$ from $\mathcal{B}$'s intrinsic metric objects. The presence of the dilaton as one of these intrinsic objects, though, allows us to construct an almost infinite variety of counterterms; and there is *not* only one expression among them which is univocally determined by requiring finite charges. We decide to consider the counterterm whose form is the simplest:

$$\underline{\mathfrak{L}} \stackrel{\text{def}}{=} 4\kappa \int_{\mathcal{B}} \sqrt{-\gamma}\, \lambda\eta; \tag{3.75}$$

it differs from the corresponding non-dilatonic one (3.61) by a factor 2 and by the presence of the dilaton. Its variation is:

$$\delta \underline{\mathfrak{L}} = \int_{\mathcal{B}} \sqrt{-\gamma}\,(2\kappa\lambda\eta\gamma^{ab}\delta\gamma_{ab} + 4\kappa\lambda\delta\eta), \tag{3.76}$$

whence we have:

$$\underline{\Pi}^{ab} \stackrel{\text{def}}{=} \frac{\delta \underline{\mathfrak{L}}}{\delta\gamma_{ab}} = 2\kappa\lambda\gamma^{ab}, \tag{3.77a}$$

$$\underline{\Pi}^{\eta} \stackrel{\text{def}}{=} \frac{\delta \underline{\mathfrak{L}}}{\delta\eta} = 4\kappa\lambda, \tag{3.77b}$$

so that the quasilocal stress-energy tensor is:

$$\begin{aligned} \tau^{ab} &= 2(\Pi_{\text{cl}}{}^{ab} - \underline{\Pi}^{ab}) \\ &= 2\kappa[\eta(\Theta^{ab} - \Theta\gamma^{ab}) + n^{\mu}\boldsymbol{\nabla}_{\mu}\eta - 2\lambda\eta\gamma^{ab}]. \end{aligned} \tag{3.78a}$$

The formula for the charge is, as from Sec. 2.3.2,

$$\mathfrak{Q}_{\mathcal{P}_t}(\xi) \stackrel{\text{def}}{=} \int_{\mathcal{P}_t} \sqrt{\boldsymbol{\sigma}}\,\xi^{\parallel a}\tilde{u}^{b}\tau_{ab}. \tag{3.78b}$$

Charges

As in Sec. 3.1.4, the ground state is:

$$ds^2 = -\lambda^2 r^2 dt^2 + (\lambda^2 r^2 + 1)^{-1} dr^2 + r^2 d\phi^2, \tag{3.79a}$$

$$\eta = \lambda r, \tag{3.79b}$$

and the excited states are:

$$ds^2 = -\left(\lambda^2 r^2 - \frac{\alpha^2}{r}\right) dt^2 + \left(\lambda^2 r^2 - \frac{\alpha^2}{r}\right)^{-1} dr^2 + r^2 d\phi^2, \tag{3.80a}$$

$$\eta = \lambda r; \tag{3.80b}$$

these are contained in the following asymptotic conditions:

$$g_{tt} = -\lambda^2 r^2 + O\left(\frac{1}{r}\right), \tag{3.81a}$$

$$g_{t\phi} = O\left(\frac{1}{r}\right), \tag{3.81b}$$

$$g_{tr} = O\left(\frac{1}{r^4}\right), \tag{3.81c}$$

$$g_{\phi\phi} = r^2 + O\left(\frac{1}{r}\right), \tag{3.81d}$$

$$g_{\phi r} = O\left(\frac{1}{r^4}\right), \tag{3.81e}$$

$$g_{rr} = \frac{1}{\lambda^2 r^2} + O\left(\frac{1}{r^5}\right), \tag{3.81f}$$

$$\eta = O(r), \tag{3.81g}$$

whose asymptotic symmetry group is just $SO(2,2)$, generated by:

$$A_1 = \left[\frac{1}{\lambda} + O\left(\frac{1}{r^4}\right)\right]\frac{\partial}{\partial t} + \left[O\left(\frac{1}{r^4}\right)\right]\frac{\partial}{\partial \phi} + \left[O\left(\frac{1}{r}\right)\right]\frac{\partial}{\partial r}, \tag{3.82a}$$

$$A_2 = \left[O\left(\frac{1}{r^4}\right)\right]\frac{\partial}{\partial t} + \left[1 + O\left(\frac{1}{r^4}\right)\right]\frac{\partial}{\partial \phi} + \left[O\left(\frac{1}{r}\right)\right]\frac{\partial}{\partial r}, \tag{3.82b}$$

$$A_3 = \left[\frac{\phi}{\lambda} + O\left(\frac{1}{r^4}\right)\right]\frac{\partial}{\partial t} + \left[\lambda t + O\left(\frac{1}{r^4}\right)\right]\frac{\partial}{\partial \phi} + \left[O\left(\frac{1}{r}\right)\right]\frac{\partial}{\partial r}, \tag{3.82c}$$

$$A_4 = \left[t + O\left(\frac{1}{r^4}\right)\right]\frac{\partial}{\partial t} + \left[\phi + O\left(\frac{1}{r^4}\right)\right]\frac{\partial}{\partial \phi} - \left[r + O\left(\frac{1}{r}\right)\right]\frac{\partial}{\partial r}, \tag{3.82d}$$

$$\begin{aligned} A_5 = &\left[\lambda t^2 + \frac{\phi^2}{\lambda} + \frac{1}{\lambda^3 r^2} + O\left(\frac{1}{r^4}\right)\right]\frac{\partial}{\partial t} + \left[2\lambda t\phi + O\left(\frac{1}{r^4}\right)\right]\frac{\partial}{\partial \phi} \\ &- \left[2\lambda t r + O\left(\frac{1}{r}\right)\right]\frac{\partial}{\partial r}, \end{aligned} \tag{3.82e}$$

$$\begin{aligned} A_6 = &\left[2t\phi + O\left(\frac{1}{r^4}\right)\right]\frac{\partial}{\partial t} + \left[\lambda^2 t^2 + \phi^2 - \frac{1}{\lambda^2 r^2} + O\left(\frac{1}{r^4}\right)\right]\frac{\partial}{\partial \phi} \\ &- \left[2\phi r + O\left(\frac{1}{r}\right)\right]\frac{\partial}{\partial r}, \end{aligned} \tag{3.82f}$$

which satisfy the commutation rules (3.54) (p. 62).

In Sec. 3.1.4 we also stated that the asymptotic conditions and symmetries above yield diverging charges; now that statement will be demonstrated. Let us express the (3.81) as:

$$g_{tt} = -\lambda^2 r^2 + \frac{\varphi_{tt}}{\lambda r} + O\left(\frac{1}{r^2}\right), \tag{3.83a}$$

$$g_{t\phi} = \frac{\varphi_{t\phi}}{\lambda r} + O\left(\frac{1}{r^2}\right), \tag{3.83b}$$

$$g_{tr} = \frac{\varphi_{tr}}{\lambda^4 r^4} + O\left(\frac{1}{r^5}\right), \tag{3.83c}$$

$$g_{\phi\phi} = r^2 + \frac{\varphi_{\phi\phi}}{\lambda r} + O\left(\frac{1}{r^2}\right), \tag{3.83d}$$

$$g_{\phi r} = \frac{\varphi_{\phi r}}{\lambda^4 r^4} + O\left(\frac{1}{r^5}\right), \tag{3.83e}$$

$$g_{rr} = \frac{1}{\lambda^2 r^2} + \frac{\varphi_{rr}}{\lambda^5 r^5} + O\left(\frac{1}{r^6}\right), \tag{3.83f}$$

$$\eta = \lambda r + \varphi_1^\eta \lambda r + \frac{\varphi_2^\eta}{\lambda r} + O\left(\frac{1}{r^2}\right), \tag{3.83g}$$

where the improper gauge part is now visible; using the expressions above into Eqs. (3.78) to calculate the charge associated to the generator $\partial/\partial t$, we obtain:

$$\mathfrak{Q}_{\mathcal{P}_t}\left(\frac{\partial}{\partial t}\right) = 4\kappa\lambda r \int_0^{2\pi} \varphi_2^\eta(t,\phi)\, d\phi + O(r^0). \tag{3.84}$$

It is now evident that the charge diverges as $r \to \infty$, and that this is due to the asymptotic condition $\eta = O(r)$. The divergence remains hidden when we calculate the charge for a black-hole solution (3.80), because $\varphi_2^\eta(t,\phi) = 0$ for such a solution. In order to have finite charges we must require the dilaton to behave as

$$\eta = \lambda r + \varphi_1^\eta \lambda r + O\left(\frac{1}{r^2}\right); \tag{3.85}$$

this requirement reduces the asymptotic symmetry group to that spanned by $\{A_1, A_2, A_3\}$: this is just the 'symmetry breaking' phenomenon we have already spoken about. While in two dimensions this phenomenon leads to non-conserved charges (as we saw by Regge and Teitelboim's formalism and as we shall see by the quasilocal formalism), so that one can keep the larger symmetry group, in three dimensions the phenomenon leads to diverging charges instead, so that one must reduce the group.

3.2.3 The two-dimensional dilatonic case

The calculation of the charges through Brown and York's method, in the two-dimensional case, is the easiest computationally yet the subtlest and the most ambiguous at the same time.

The counterterm question

By detailed examination of charge calculation in two-dimensional dilatonic anti-de Sitter space, all conceptual problems concerning the counterterm are brought to light. We shall calculate the charges both through an intrinsic counterterm and through a background counterterm. We shall see that a counterterm *à la* Balasubramanian and Kraus cannot be univocally determined by simply demanding renormalised charges. We shall also see that Brown and York's formula for the charges is not equivalent to the Regge-Teitelboim one.

Intrinsic counterterm

We use an intrinsic counterterm which is the sum of two pieces:

1. the first is the analogue of the one used in the three-dimensional dilatonic case:

$$\underline{\mathfrak{L}}_1 \stackrel{\text{def}}{=} 2C_1\kappa \int_{\mathcal{B}} \sqrt{-\gamma}\, \lambda\eta; \tag{3.86}$$

2. for the second, we draw inspiration from the dilaton kinetic term in the Brans-Dicke Lagrangian:

$$\underset{\sim}{\mathfrak{L}}_2 \stackrel{\text{def}}{=} C_2\kappa \int_{\mathcal{B}} \sqrt{-\gamma}\,\frac{1}{\lambda\eta}(\boldsymbol{\Delta}\eta)^2; \tag{3.87}$$

so the total intrinsic counterterm is:

$$\underset{\sim}{\mathfrak{L}} \stackrel{\text{def}}{=} \kappa \int_{\mathcal{B}} \sqrt{-\gamma}\left[2C_1\lambda\eta + C_2\frac{1}{\lambda\eta}(\boldsymbol{\Delta}\eta)^2\right] = \underset{\sim}{\mathfrak{L}}_1 + \underset{\sim}{\mathfrak{L}}_2. \tag{3.88}$$

We shall see that the first addendum takes part in charge renormalisation, while the other appears in the finite part of the charges. The constants C_1 and C_2 are undetermined by now; according to Balasubramanian and Kraus we should be able to fix them by requiring renormalised charges; we shall see that this is not the case.

The variation of the counterterm is:

$$\begin{aligned}
\delta\underset{\sim}{\mathfrak{L}} = \kappa \int_{\mathcal{B}} \sqrt{-\gamma}\Bigg\{ &\bigg[C_1\lambda\eta\gamma^{ab} \\
&+ C_2\frac{1}{2\lambda\eta}(\gamma^{ab}\gamma^{cd} - 2\gamma^{ac}\gamma^{bd})\boldsymbol{\Delta}_c\eta\,\boldsymbol{\Delta}_d\eta\bigg]\delta\gamma_{ab} \\
&+ \bigg[2C_1\lambda - C_2\frac{1}{\lambda\eta^2}\gamma^{cd}\boldsymbol{\Delta}_c\eta\,\boldsymbol{\Delta}_d\eta \\
&- 2C_2\partial_c\bigg(\sqrt{-\gamma}\,\frac{1}{\lambda\eta}\gamma^{cd}\boldsymbol{\Delta}_d\eta\bigg)\bigg]\delta\eta\Bigg\};
\end{aligned} \tag{3.89}$$

and we find the expression for the quasilocal stress-energy tensor through Eq. (2.66):

$$\begin{aligned}
\tau^{ab} = 2\kappa[&\eta(\Theta^{ab} - \Theta\gamma^{ab}) + n^\mu\boldsymbol{\nabla}_\mu\eta - C_1\lambda\eta\gamma^{ab} \\
&- C_2\frac{1}{2\lambda\eta}(\gamma^{ab}\gamma^{cd} - 2\gamma^{ac}\gamma^{bd})\boldsymbol{\Delta}_c\eta\,\boldsymbol{\Delta}_d\eta].
\end{aligned} \tag{3.90}$$

The charge associated to the generator ξ is, as usual,

$$\mathfrak{Q}_{\mathcal{P}}(\xi) \stackrel{\text{def}}{=} \int_{\mathcal{P}_t} \sqrt{\boldsymbol{\sigma}}\,\xi^{\parallel a}\tilde{u}^b\tau_{ab}. \tag{3.91}$$

Charges We already saw in Sec. 3.1.3 that the asymptotic conditions for two-dimensional anti-de Sitter space are:

$$g_{tt} = -\lambda^2 r^2 + O(r^0), \tag{3.92a}$$

$$g_{tr} = O\left(\frac{1}{r^3}\right), \tag{3.92b}$$

$$g_{rr} = \frac{1}{\lambda^2 r^2} + O\left(\frac{1}{r^4}\right), \tag{3.92c}$$

$$\eta = O(r); \tag{3.92d}$$

these are invariant under the conformal group in one dimension, whose generators are given by:

$$\xi = \left[\varepsilon(t) + \frac{1}{2\lambda^4 r^2}\frac{d^2\varepsilon(t)}{dt^2} + O\left(\frac{1}{r^4}\right)\right]\frac{\partial}{\partial t} + \left[r\varepsilon(t) + O\left(\frac{1}{r}\right)\right]\frac{\partial}{\partial r}, \tag{3.93}$$

or, by means of a countable basis,

$$\begin{aligned} A_n = A_{-n} = &\left[\frac{1}{\lambda}\left(1 - \frac{n^2}{2\lambda^2 r^2}\right)\cos(n\lambda t) + O\left(\frac{1}{r^4}\right)\right]\frac{\partial}{\partial t} \\ &+ \left[rn\sin(n\lambda t) + O\left(\frac{1}{r}\right)\right]\frac{\partial}{\partial r}, \end{aligned} \tag{3.94a}$$

$$\begin{aligned} B_n = -B_{-n} = &\left[\frac{1}{\lambda}(1 - \frac{n^2}{2\lambda^2 r^2}\right)\sin(n\lambda t) + O\left(\frac{1}{r^4}\right)\Bigg]\frac{\partial}{\partial t} \\ &+ \left[rn\cos(n\lambda t) + O\left(\frac{1}{r}\right)\right]\frac{\partial}{\partial r}. \end{aligned} \tag{3.94b}$$

The generators form a Virasoro algebra and satisfy the commutation relations (3.39) (p. 59).

In order to analyse in detail the calculations to follow, we write the asymptotic conditions (3.92) as:

$$g_{tt} = -\lambda^2 r^2 + \varphi_{tt} + O\left(\frac{1}{r}\right), \tag{3.95a}$$

$$g_{tr} = \frac{\varphi_{tr}}{\lambda^3 r^3} + O\left(\frac{1}{r^4}\right), \tag{3.95b}$$

$$g_{rr} = \frac{1}{\lambda^2 r^2} + \frac{\varphi_{rr}}{\lambda^4 r^4} + O\left(\frac{1}{r^5}\right), \tag{3.95c}$$

$$\eta = \lambda r\rho + \frac{\varphi_{\phi\phi}}{\lambda r} + O\left(\frac{1}{r^2}\right), \tag{3.95d}$$

where the $O(1/r^n)$ terms are just the proper gauge parts. In the notation above, the ground state corresponds to

$$\varphi_{tt} = \varphi_{tr} = \varphi_{rr} = \varphi_{\phi\phi} = 0, \tag{3.96a}$$

$$\rho = 1, \tag{3.96b}$$

while the conditions

$$\varphi_{tt} = \varphi_{rr} = \frac{2M}{\lambda}, \tag{3.97a}$$

$$\varphi_{tr} = \varphi_{\phi\phi} = 0, \tag{3.97b}$$

$$\rho = 1 \tag{3.97c}$$

correspond to a black hole having mass M.

Firstly, we want to examine the asymptotic behaviour of the term

$$\sqrt{\sigma}\,\xi^{\|a}\tilde{u}^b \equiv \xi^{\|t}\tilde{u}^t \tag{3.98}$$

which multiplies the quasilocal tensor in Eq. (3.91). We find:

$$\xi^{\|t}\tilde{u}^t \equiv \xi^\mu \gamma_\mu{}^t \tilde{u}^t = \frac{\varepsilon}{\lambda r} + O\left(\frac{1}{r^3}\right); \tag{3.99}$$

it is clear that, if we want non-diverging, yet non-vanishing charges, the quasilocal tensor must behave exactly as $O(r)$; the analysis of his asymptotic behaviour gives:

$$\begin{aligned}\tau_{tt} &= 2\kappa[(C_1-1)\lambda^4\rho]\cdot r^3 \\ &\quad + \kappa\left[\rho\varphi_{rr} - 2(C_1-1)\rho\varphi_{tt} + 2(C_1+1)\varphi_{\phi\phi} + C_2\frac{\dot{\rho}^2}{\lambda^2\rho}\right]\cdot r;\end{aligned} \tag{3.100}$$

in this expression we have a term which behaves as $O(r)$, which is just what we wanted, but also a term

$$2\kappa(C_1-1)\lambda^4\rho r^3 \sim O(r^3) \tag{3.101}$$

which yields a divergence. We get rid of it by fixing the constant C_1 to the suitable value

$$C_1 = 1. \tag{3.102}$$

We have thus fixed one of the counterterm's constants; the expression for the quasilocal tensor becomes:

$$\tau_{tt} = \kappa\left(\rho\varphi_{rr} + 4\varphi_{\phi\phi} + C_2\frac{\dot{\rho}^2}{\lambda^2\rho}\right)r; \tag{3.103}$$

Now, multiplying Eq. (3.99) by Eq. (3.100) with $C_1 = 1$, we find the expression for the charge (integration over $\mathcal{P}$ corresponds to evaluation at $\mathcal{P}$, since $\mathcal{P}$ is just a point in this case):

$$\mathfrak{Q}(\xi) = \kappa\lambda\varepsilon\left(\rho\varphi_{rr} + 4\varphi_{\phi\phi} + C_2\frac{\dot{\rho}^2}{\lambda^2\rho}\right). \tag{3.104}$$

Calculation of the charge associated to the Killing vector field $\partial/\partial t$ (Eq. (3.93) with $\varepsilon(t)\equiv 1$) for a black-hole solution with mass M (Eqs. (3.97)), yields:

$$\mathfrak{Q}\left(\frac{\partial}{\partial t}\right) = 2\kappa \equiv M \tag{3.105}$$

(since $\kappa = \frac{1}{2}$ for the two-dimensional Lagrangian, q.v. Sec. 1.3.2).

Thus Eq. (3.104) gives the right value for the black-hole mass, independently of the value of the constant C_2. Note that we do not have other reasonable requirements to impose on the expression of the charge, and so the constant C_2 remains hopelessly undetermined; Balasubramanian and Kraus' prescription is ambiguous in this case.

We want to check the values of the central charges. Let us take the infinitesimal generator:

$$\epsilon\zeta = \left[\epsilon\omega(t) + \frac{1}{2\lambda^4 r^2}\epsilon\ddot{\omega}(t) + O\left(\frac{1}{r^4}\right)\right]\frac{\partial}{\partial t} + \left[r\epsilon\dot{\omega}(t) + O\left(\frac{1}{r}\right)\right]\frac{\partial}{\partial r}, \quad (3.106)$$

and deform the ground state by: $g_{\mu\nu} + \mathbf{L}_{\epsilon\zeta} g_{\mu\nu}$ and $\eta + \mathbf{L}_{\epsilon\zeta}\eta$; we find that this infinitesimally deformed state is given by Eqs. (3.95) with:

$$\varphi_{tt} = -\frac{\epsilon\dddot{\omega}}{\lambda^2}, \quad (3.107a)$$

$$\varphi_{tr} = \varphi_{rr} = \varphi_{\phi\phi} = 0, \quad (3.107b)$$

$$\rho = 1 - \epsilon\dot{\omega}. \quad (3.107c)$$

Substitution in Eq. (3.104) of the Equations above gives the central charge associated to the generator ξ; at first order in ϵ:

$$\mathfrak{Q}(\xi) = 0 + O(\epsilon^2). \quad (3.108)$$

This result is quite disturbing: we do not find Cadoni and Mignemi's result, Eq. (3.45). A discussion about this incongruence is deferred to the next section and to the last chapter.

Background counterterm

The main difficulty in calculating a background counterterm consists in embedding *isometrically* and, in this case, *isodilatonically* the hypersurface $\mathcal{B}$ in the reference spacetime.

In the present case the hypersurface $\mathcal{B}$ is a unidimensional manifold (i.e. a line) given by the equation $r =$ const., and its intrinsic metric and dilaton are given by:

$$\gamma_{tt} = -\lambda^2 r^2 + \varphi_{tt}(t) + O\left(\frac{1}{r}\right), \quad (3.109a)$$

$$\eta|_{\mathcal{B}} = \lambda r\rho + \frac{\varphi_{\phi\phi}}{\lambda r} + O\left(\frac{1}{r^2}\right); \quad (3.109b)$$

the background spacetime is defined by:

$$ds^2 = -\lambda^2 r^2 dt^2 + (\lambda^2 r^2)^{-1} dr^2, \quad (3.110a)$$

$$\eta = \lambda r, \quad (3.110b)$$

and in this coordinate system the metric and the dilaton induced on the boundary $r =$ const. are:

$$\gamma_{tt} = -\lambda^2 r^2, \quad (3.111a)$$

$$\eta|_{\mathcal{B}} = \lambda r; \quad (3.111b)$$

so they do not coincide with Eqs. (3.109). We must find — if it is possible — a new coordinate system for the reference spacetime so that the boundary metric and dilaton coincide, asymptotically at least, with the ones given by Eqs. (3.109).

Such a coordinate system exists, and is obtained through the transformation:

$$t \longmapsto \tau(t) + \frac{\sigma(t)}{\lambda^2 r^2}, \tag{3.112a}$$

$$r \longmapsto r\rho(t) + \frac{\varphi_{\phi\phi}(t)}{\lambda^2 r}, \tag{3.112b}$$

where the functions $\tau(t)$ and $\sigma(t)$ are defined by:

$$\tau = \varsigma \int dt\, \rho^{-1}, \tag{3.112c}$$

$$\sigma = \varsigma \int dt \left(\frac{\dot{\rho}^2}{2\lambda^2 \rho^3} - \frac{\varphi_{tt}}{2\rho} - \frac{\varphi_{\phi\phi}}{\rho^2} \right), \tag{3.112d}$$

with $\varsigma = \pm 1$.

The line element and the dilaton (3.110) assume thus the new forms:

$$\begin{aligned} ds^2 = {} & \left[-\lambda^2 r^2 + \varphi_{tt} + O\left(\frac{1}{r^2}\right) \right] dt^2 + 2\left[\frac{\check{\varphi}_{tr}}{\lambda r} + O\left(\frac{1}{r^3}\right) \right] dt\, dr \\ & + \left[\frac{1}{\lambda^2 r^2} + \frac{\check{\varphi}_{rr}}{\lambda^4 r^4} + O\left(\frac{1}{r^5}\right) \right] dr^2, \end{aligned} \tag{3.113a}$$

$$\eta = \lambda r \rho + \frac{\varphi_{\phi\phi}}{\lambda r}, \tag{3.113b}$$

where

$$\check{\varphi}_{rr} \stackrel{\text{def}}{=} -4\lambda^2 \rho^2 \sigma^2 - 4\frac{\varphi_{\phi\phi}}{\rho}, \tag{3.113c}$$

$$\check{\varphi}_{tr} \stackrel{\text{def}}{=} 2\varsigma\lambda\rho\sigma + \frac{\dot{\rho}}{\lambda\rho}; \tag{3.113d}$$

in this new coordinate system, the metric and dilaton induced on the hypersurface $r = \text{const.}$ have expressions which asymptotically coincide with Eqs. (3.109) (to be precise, their differences affect the proper gauge parts only):

$$\gamma_{tt} = -\lambda^2 r^2 + \varphi_{tt}(t) + O\left(\frac{1}{r^2}\right), \tag{3.114a}$$

$$\eta|_{\mathcal{B}} = \lambda r \rho + \frac{\varphi_{\phi\phi}}{\lambda r}. \tag{3.114b}$$

From the definition

$$\underset{\sim}{\mathfrak{L}} \stackrel{\text{def}}{=} \int_{\mathcal{B}} (\tilde{N} \underset{\sim}{\boldsymbol{E}} - \tilde{N}^A \underset{\sim}{\boldsymbol{J}}_A), \tag{3.115}$$

we obtain, through simple calculations:

$$\underline{\Pi}_{tt} = \kappa\left[-\lambda^4 r^3 \rho + \lambda^2 r\left(\rho\varphi_{tt} - \varphi_{\phi\phi} - \frac{\dot{\rho}^2}{2\lambda^2\rho}\right)\right] + O(r^0); \tag{3.116}$$

hence the full quasilocal tensor is given by:

$$\tau_{tt} = \kappa\left(\rho\varphi_{rr} + 4\varphi_{\phi\phi} + \frac{\dot{\rho}^2}{\lambda^2\rho}\right)r. \tag{3.117}$$

Charges Comparison with the quasilocal tensor constructed with the intrinsic counterterm shows that the latter coincides with the present, background one when $C_2 = 1$. Since the constant C_2 did not appear in the main results concerning the charges that we obtained through the use of the intrinsic counterterm, we can conclude that an analysis of the charges through the present counterterm would just lead to the same results. So we cannot find Cadoni and Mignemi's result for the central charge in this case either. The only explanation for this incongruence is that Brown and York's approach to charge analysis is not equivalent to Regge and Teitelboim's. In particular, the surface terms associated with these two approaches, Eqs. (2.35) and (2.36), are not equivalent.

3.3 Central charges and statistical entropy

3.3.1 Central charge

The fact that a non-dilatonic gravity theory in three-dimensional anti-de Sitter space possesses an infinite-dimensional (asymptotic) symmetry group with associated charges holds many interesting consequences for the statistical thermodynamics of the theory.

The algebra of the asymptotic symmetry canonical generators usually yields a representation of the Lie algebra of the corresponding vector fields. This representation is, in general, a projective one:

$$\{\mathfrak{H}[\xi], \mathfrak{H}[\zeta]\} = \mathfrak{H}[\,[\xi, \zeta]\,] + K[\xi, \zeta]; \tag{3.118}$$

in many cases, however, one has $K[\cdot, \cdot] = 0$ and Eq. (3.118) is just an isomorphism:

$$\{\mathfrak{H}[\xi], \mathfrak{H}[\zeta]\} = \mathfrak{H}[\,[\xi, \zeta]\,]. \tag{3.119}$$

The functional $K[\cdot, \cdot]$ is called 'central charge'.

Eq. (3.119) does hold in the case of canonical gravity theory in flat space, but does not in the case of anti-de Sitter space, for which we need the more general projective representation. The calculation of the explicit value of the central charge has been studied by Brown and Henneaux [14].

Starting from Eq. (3.118) one first notes that, for solutions of the equations of motion, it becomes:

$$\{\mathfrak{J}[\xi], \mathfrak{J}[\zeta]\} = \mathfrak{J}[\,[\xi, \zeta]\,] + K[\xi, \zeta]; \tag{3.120}$$

moreover, the Dirac bracket of two canonical generators $\{\mathfrak{J}[\xi], \mathfrak{J}[\zeta]\}$ is given by the variation of the charge associated to the generator ξ on the surface deformed by ζ:

$$\{\mathfrak{J}[\xi], \mathfrak{J}[\zeta]\} \equiv \delta_\zeta \mathfrak{J}[\xi]; \tag{3.121}$$

hence Eq. (3.120) can be rewritten as:

$$\delta_\zeta \mathfrak{J}[\xi] = \mathfrak{J}[\,[\xi, \zeta]\,] + K[\xi, \zeta], \tag{3.122}$$

and when evaluated on the ground state, for which $\mathfrak{J}[\cdot] = 0$, it reduces to:

$$\delta_\zeta \mathfrak{J}[\xi] = K[\xi, \zeta]. \tag{3.123}$$

Therefore we have that the charges evaluated in the preceding sections are just the central charges of the projective representation of the asymptotic symmetry generators' algebra. We have seen that in the case of three-dimensional anti-de Sitter space without dilaton and two-dimensional anti-de Sitter space with dilaton these algebras are infinite-dimensional.

The three-dimensional case

Using a new basis for the algebra of generators $\{A_n, B_n, C_n, D_n\}$ (Eq. (3.25)):

$$L_n \stackrel{\text{def}}{=} \frac{i\sigma}{2} A_n + \frac{i\sigma}{2} B_n - \frac{1}{2} C_n + \frac{1}{2} D_n - \frac{1}{\lambda}, \tag{3.124a}$$

$$K_n \stackrel{\text{def}}{=} \frac{i\sigma}{2} A_n - \frac{i\sigma}{2} B_n - \frac{1}{2} C_n - \frac{1}{2} D_n - \frac{1}{\lambda}, \tag{3.124b}$$

we obtain the following commutation rules for the corresponding canonical generators:

$$\{\mathfrak{J}[L_n], \mathfrak{J}[L_m]\} = (n-m)\mathfrak{J}[L_{(n+m)}] + \frac{c}{12}(n^3 - n)\delta_{n,-m}, \tag{3.125a}$$

$$\{\mathfrak{J}[K_n], \mathfrak{J}[K_m]\} = (n-m)\mathfrak{J}[K_{(n+m)}] + \frac{c}{12}(n^3 - n)\delta_{n,-m}, \tag{3.125b}$$

$$\{\mathfrak{J}[L_n], \mathfrak{J}[K_m]\} = 0, \tag{3.125c}$$

with $c \equiv \frac{3}{2\lambda}$. The fundamental result is that *the algebra of the canonical asymptotic symmetry generators in three-dimensional anti-de Sitter space is a direct sum of two Virasoro algebras with central charge* $c \equiv \frac{3}{2\lambda}$. This algebra characterizes a conformal field theory in two dimensions, hence *gravity theory in three-dimensional anti-de Sitter space is dual to a conformal field theory in two dimensions with central charge* $c \equiv \frac{3}{2\lambda}$.

The two-dimensional case

The asymptotic symmetry group in two-dimensional anti-de Sitter space, instead, is generated by $\{A_n, B_n\}$, Eq. (3.38), and, by a change of basis:

$$L_n \stackrel{\text{def}}{=} iA_n - B_n - 1, \tag{3.126}$$

we have the following expressions for the commutation rules of the corresponding canonical generators:

$$\{\mathfrak{J}[L_n], \mathfrak{J}[L_m]\} = (n-m)\mathfrak{J}[L_{(n+m)}] + \frac{c}{12}(n^3 - n)\delta_{n,-m}, \tag{3.127}$$

with $c \equiv 24$, as calculated by Cadoni and Mignemi [19]. Hence *the algebra of the canonical asymptotic symmetry generators in two-dimensional anti-de Sitter space is a Virasoro algebra with central charge* $c \equiv 24$. This algebra characterizes a conformal field theory in one dimension, so that *gravity theory in two-dimensional anti-de Sitter space is dual to a conformal field theory in one dimension with central charge* $c \equiv 24$.[2]

3.3.2 Statistical entropy

The conclusions in the preceding section imply that a black-hole solution in three- or two-dimensional anti-de Sitter space (without dilaton and with dilaton, respectively) can be considered as a state (an excited one) of a conformal dual theory. From this point of view, the entropy of a black hole having mass M (and angular momentum J in three dimensions) can be statistically evaluated by counting the corresponding microstates. As it has been shown by Strominger [42], when the number of the states tends to infinity the entropy in a conformal field theory is given by Cardy's formula [22]

$$S = 2\pi\sqrt{\frac{c\, l_0^L}{6}} + 2\pi\sqrt{\frac{c\, l_0^K}{6}} \tag{3.128}$$

in two dimensions, or

$$S = 2\pi\sqrt{\frac{c\, l_0^L}{6}} \tag{3.129}$$

in one dimension, where l_0^L (l_0^K) is the eigenvalue of the generator L_0 (K_0), and the formula holds for high values of mass and angular momentum.

In the three-dimensional case we have:

$$S = \pi\sqrt{\frac{M}{2\lambda^2} + \frac{J}{2\lambda}} + \pi\sqrt{\frac{M}{2\lambda^2} - \frac{J}{2\lambda}}, \tag{3.130}$$

and in the two-dimensional case:

$$S = 4\pi\sqrt{\frac{M}{\lambda}}. \tag{3.131}$$

The first value agrees with the thermodynamic one, as found by Bekenstein and Hawking, and this agreement supports the interpretation of three-dimensional gravity theory as a two-dimensional conformal field theory. The second value,

[2] *Note added in translation:* See Note 3

instead, does not agree with the thermodynamic one by a factor $\sqrt{2}$. A possible explanation for this discrepancy may come from the fact that we considered only one of the two disconnected pieces of the two-dimensional anti-de Sitter space's boundary — we were forced to do that by the dilaton's presence. Another possible explanation is that Hamiltonian and Lie evolution do not coincide in the two-dimensional theory, as it happens in the three-dimensional one instead.[3]

[3] *Note added in translation:* Cadoni, Carta, Klemm, and Mignemi [*AdS_2 Gravity as a Conformally Invariant Mechanical System*, preprint hep-th/0009185 (2000)] have recently shown that two-dimensional anti-de Sitter space is really dual to a conformally invariant theory that can be described in terms of a de Alfaro-Fubini-Furlan model coupled to an external source with conformal dimension two, or equivalently in terms of a mechanical system with anholonomic constraints. They have found perfect agreement between statistical and thermodynamic entropy; the discrepancy above vanishes thanks to an entropy contribution which describes the entanglement of states.

Chapter 4

Final remarks and conclusions

The main results of the present work can be grouped as follows:

1. derivation and discussion of the asymptotic symmetries of three-dimensional anti-de Sitter space for a Jackiw-Teitelboim dilaton gravity theory;

2. application of Brown and York's quasilocal formalism to the calculation of the charges associated to asymptotic symmetries in two and three dimensions;

3. comparative discussion, in the context of the asymptotic symmetries, of the Hamiltonian surface terms which have recently appeared in the literature.

4.1 Asymptotic symmetries in three-dimensional anti-de Sitter space

Jackiw-Teitelboim dilaton gravity theory in three-dimensional anti-de Sitter space has proved to be very different from the non-dilatonic one in the context of the asymptotic symmetries. The presence of a dilaton field has three main consequences.

Firstly, it introduces a dynamical degree of freedom into the theory, that would have none otherwise.

Secondly, the Jackiw-Teitelboim black-hole solution is very different from the Bañados-Teitelboim-Zanelli one: the second is *topological* because the scalar curvature is everywhere constant (q.v. Sec. 1.3.2), whereas the first is not because the curvature is not constant; hence the second has got a simple causal singularity, whereas the first has got a polynomial one. These topological and causal differences manifest themselves as different asymptotic conditions of the two metrics: the proper gauge parts in the dilaton theory fall off faster than in the non-dilatonic theory by one power of $1/r$; as a consequence, the asymptotic group is smaller and finite-dimensional: it is just the Special Orthogonal group $SO(2,2)$. All this yields the conjecture that the possibility of an infinite-dimensional extension of the symmetries should depend upon one, or the union of some, of the following three points: the absence of dynamical degrees of freedom; the presence of topological solutions; the absence of polynomial singularities.

Thirdly, the asymptotic condition for the dilaton yields diverging charges, so that it must be modified and this leads to the breaking of the symmetries and to further reduction of the $SO(2,2)$ group.

Symmetry breaking and dilaton

We saw that the symmetry breaking phenomenon due to the presence of a (non-constant) dilaton field appears in the two-dimensional case as well. This fact shows up clearly in the expression for the charge associated to the generator ξ^μ for a Jackiw-Teitelboim-like theory,

$$-[\mathfrak{Q}_{\mathcal{P}_{t''}}(\xi) - \mathfrak{Q}_{\mathcal{P}_{t'}}(\xi)] = \tag{4.1}$$

$$\int_{t'}^{t''} \sqrt{-\gamma}\,[(\Pi^\eta{}_{\rm cl} - \underline{\Pi}^\eta)\xi^a \boldsymbol{\Delta}_a \eta - \xi^a \gamma_{a\mu} n_\nu T^{\mu\nu}], \tag{4.2}$$

and in the equation for the quasilocal tensor,

$$\boldsymbol{\Delta}_b \tau^{ab} = (\Pi^\eta{}_{\rm cl} - \underline{\Pi}^\eta)\boldsymbol{\Delta}_b \eta \gamma^{ab} - \gamma^b{}_\mu n_\nu T^{\mu\nu}, \tag{4.3}$$

where source terms appear whose origin is manifestly dilatonic; these additional source terms force the imposition of additional constraints on the symmetry generators, that are reduced in number this way. If one disregards these additional constraints, the dilatonic source terms lead to non-conserved charges (as is the case for two-dimensional anti-de Sitter space) or, worst, diverging charges (as is the case for three-dimensional anti-de Sitter space); in the latter case one cannot actually disregard the additional constraints, and the breaking of the symmetries is inescapable.

4.2 Quasilocal formalism and asymptotic symmetries

Applying Brown and York's formalism to the calculation of the asymptotic-symmetry-associated charges, we have obtained results which do not agree with those calculated by Regge and Teitelboim's prescription in the two-dimensional case (Sec. 3.1.3 and Sec. 3.2.3): whereas the Regge-Teitelboim procedure yields non-vanishing (non-conserved) central charges, the quasilocal formalism yields vanishing charges. There are two possible explanations for this discrepancy.

The first is that the Regge-Teitelboim Hamiltonian surface term,

$$
\begin{aligned}
\mathfrak{J}_{\mathrm{RT}}[\xi] \stackrel{\text{def}}{=} & \text{‘integration of} \\
& \int_{\mathcal{P}} \frac{\sqrt{\boldsymbol{\sigma}}}{\sqrt{\boldsymbol{h}}} \tilde{n}_l \{ \kappa \boldsymbol{G}^{ijkl} [N\eta \mathbf{D}_k h_{ij} - \partial_k (N\eta)\, \delta h_{ij}] \\
& \quad + \kappa \sqrt{\boldsymbol{h}}\, (2h^{il} h^{jk} - h^{ij} h^{kl}) \delta h_{ij}\, N\, \partial_k \eta \\
& \quad + 2\kappa \sqrt{\boldsymbol{h}}\, (\partial_j N\, \delta\eta - N \partial_j\, \delta\eta) \\
& \quad + (2N^i \boldsymbol{P}^{kl} - N^l \boldsymbol{P}^{ik}) \delta h_{ik} \\
& \quad + 2N_i\, \delta \boldsymbol{P}^{il} - N^l \boldsymbol{P}^{\eta}\, \delta\eta \}\text{’},
\end{aligned}
\tag{4.4}
$$

with $\xi^\mu = N u^\mu + N^\mu$, is not equivalent to Brown and York's formula for the charge, which is just Creighton and Mann's Hamiltonian surface term:

$$
\mathfrak{Q}(\xi) \stackrel{\text{def}}{=} \int_{\mathcal{P}} \sqrt{\boldsymbol{\sigma}}\, \xi^a \tilde{u}^b \tau_{ab} \equiv \mathfrak{Q}(\xi) \equiv \int_{\mathcal{P}} (\tilde{N} \boldsymbol{E} - \tilde{N}^A \boldsymbol{J}_A), \tag{4.5a}
$$

with

$$
\boldsymbol{E} = 2\sqrt{\boldsymbol{\sigma}}\, \tilde{u}_a \tilde{u}_b \Pi^{ab} = 2\sqrt{\boldsymbol{\sigma}} \left(-n^\mu \boldsymbol{\nabla}_\mu \eta + \eta k \right) - \underline{\boldsymbol{E}}, \tag{4.5b}
$$

$$
\boldsymbol{J}_A = 2\sqrt{\boldsymbol{\sigma}}\, \sigma_{Aa} \tilde{u}_b \Pi^{ab} = 2\sqrt{\boldsymbol{\sigma}}\, \sigma_{Ai} n_j P^{ij} - \underline{\boldsymbol{J}}_A, \tag{4.5c}
$$

and $\xi^a = \tilde{N} \tilde{u}^a + \tilde{N}^a$. These two surface terms may come from Lagrangians with different boundary conditions, i.e. different boundary terms.[1]

The second explanation is that the procedure, used in the present work, of substituting the generator ξ with its *projection* onto the boundary, $\xi^{\parallel a} \equiv \gamma^a{}_\mu \xi^\mu$, in the formula for the charge is inconsistent. It is very likely for this explanation to be right: it is obvious that a projection implies some loss of information; but we should like to stress the fact that we have been forced to use such a procedure, due to a lack of generality in Brown and York's formalism and in Creighton and Mann's surface term.

The problem is that formula (4.5) requires the generator ξ^μ to lie on the boundary $\mathcal{B}$, as is shown by the expression $\xi^a = \tilde{N} \tilde{u}^a + \tilde{N}^a$ (from the point of view of the quasilocal approach, this derives from the requirement that ξ^μ be a isometry of the boundary; from the point of view of Creighton and Mann's approach, this derives from the requirement that ξ^μ make the boundary evolve tangentially to itself). But an asymptotic symmetry generator is not, in general, tangential to the boundary; hence the need for its projection.

An alternative solution to this problem could be the choice of a suitable boundary that should contain the orbits of the generator; however, this solution would have two drawbacks.

[1] *Note added in translation:* Recent calculations seem to show that the Creighton-Mann Hamiltonian surface term should correspond to the Regge-Teitelboim one with an additional term proportional to the trace of the extrinsic curvature of $\mathcal{B}$, $\int_{\mathcal{P}} \sqrt{\boldsymbol{h}}\, \tilde{N} f \Theta$.

The first drawback is that one should give up a certain computational facility, since a boundary of the form x^α = const. would not be suitable in general; moreover one should choose a different boundary for every different generator.

The second drawback is much more fundamental: it is not possible to find such a boundary — which has to be timelike — for all kinds of generators. An example may help in clarifying this point. Consider three-dimensional Minkowski space $\mathbb{M}^3$ with a cylindrical coordinate system (t, r, ϕ), and consider the following symmetry generators: $\partial/\partial t$, $\partial/\partial\phi$, $2\,\partial/\partial t + \partial/\partial r$, and $\partial/\partial r$. In order to calculate the charges associated with the first two generators, one can use the boundary defined by r = const., which contains the orbits of both; in order to calculate the charge associated with the third, the boundary above is no more suitable for it does not contain the orbits, and one must resort to a boundary defined by $t - 2r$ = const. (slightly less easy computationally). But, for the fourth generator, $\partial/\partial r$, no timelike boundary at all[2] exists that can contain its orbits. Thus Brown and York, and Creighton and Mann's methods are limited to certain kinds of generators only.

Another difficulty is closely related to the point discussed above, and concerns the relationship between boundary- and bulk-symmetries in the limit where the boundary is pushed to infinity. An example may help in explaining this point as well. Consider the manifold given by an infinitely high cylindrical portion of the above-considered Minkowski space, delimited by the bases $t = -\infty$, $t = +\infty$, and by the lateral surface r = const. This manifold is locally invariant under the full three-dimensional Poincaré group, yet this group is not admitted as a group of global symmetries, for evident reasons (e.g. a spatial translation would not map the manifold into itself); indeed the only global symmetries are spatial rotations and temporal translations, generated by $\partial/\partial\phi$ and $\partial/\partial t$ respectively. The boundary does naturally reflect the group of the manifold's *global* symmetries, i.e. it is invariant under rotations and temporal translations. But, as soon as the lateral boundary is pushed to infinity, the manifold (the bulk) suddenly acquires the full Poincaré group as group of global symmetries — just because it has become Minkowski space $\mathbb{M}^3$ —, whereas the boundary, that in the limit process has just two symmetries for every finite value of r, eventually still possesses just those two initial symmetries. This way we are facing the paradox of a surface that does not possess all the symmetries of the bulk. The paradox clearly arises in the limit process: considering a boundary at finite and then push it to infinity is not the same as having a boundary 'already' at infinity.

This question is very important in the context of the asymptotic symmetries, where the fundamental principle is the fact that the asymptotic boundary possesses all the bulk symmetries (and more).

Thus the quasilocal formalism is not completely suitable for dealing with the asymptotic symmetries and charges, for it should be applied in two successive steps: (1) study the boundary's symmetries at finite, then (2) push the boundary to infinity and calculate the charges there; but we have just seen that we cannot

[2]Which satisfies some basic requirements, like e.g. having an inside and an outside.

find all asymptotic symmetries this way. This also relates to what we said in Sec. 2.1.1 about defining the metric induced at infinity by limit or by series expansion.

4.3 Different Hamiltonian surface terms

In the present work we have used two Hamiltonian surface terms recently presented in the literature, Eqs. (4.4) and (4.5), to operate with asymptotic symmetries. The conclusions about the (inadequacy of) the latter have already been drawn in the previous section.

The Regge-Teitelboim surface term does not suffer the limitations of the Creighton-Mann one; this is shown by the equation $\xi^\mu = N u^\mu + N^\mu$, which does not require the generator ξ to be tangent to any boundary. This freedom allows computational easiness and applicability to all kinds of generators. The only drawback of the Regge-Teitelboim procedure is the fact that it is not always possible to integrate the variation so as to obtain a finite expression for the surface term, as we saw e.g. during the calculations for the two- and three-dimensional dilatonic cases.

A Hamiltonian surface term recently proposed by Hawking and Hunter, Eq. (1.27), seems not to suffer from the latter drawback, while retaining the flexibility of the Regge-Teitelboim term. Its form has only been given for non-dilatonic gravity theories at the moment, though.[3]

Anyway, one should note that all Hamiltonian surface terms do in principle share a common drawback, namely the fact that they are 'Hamiltonian'; by this we mean the following fact. Using a Hamiltonian surface term to calculate a charge means that we are *canonically* evolving the initial hypersurface $\mathcal{S}'$, which satisfies certain (fixed) asymptotic conditions. But canonical evolution does not guarantee that the initial asymptotic conditions will be satisfied by the successive hypersurfaces, for the asymptotic conditions where studied by means of Lie transport, which differs in general from Hamiltonian transport. Thus we have this vicious circle: from certain asymptotic conditions we find asymptotic symmetries whose generators does not preserve those conditions (under Hamiltonian transport). This is a typically Hamiltonian problem.

Counterterms

The calculation of the charges by means of the quasilocal formalism has allowed us to analyse two different kinds of counterterm used in the literature: the background-space counterterm and the intrinsic counterterm.

The first does not yields anomalous results at finite (q.v. Sec. 2.3.1), but cannot be computed, in principle, in all cases.

The second does yield anomalous results at finite instead, and in the case of a dilaton gravity theory it cannot be univocally determined by renormalisation conditions (q.v. Sec. 3.2.3 and the indetermination of the constant C_2).

[3] *Note added in translation:* See Chap. 1., Note 1.

A sort of conclusion

Whence, exceptionally, one draws nothing

[In the original, Italian version of the present thesis, this section is an excerpt from Robert Musil's *Der Mann ohne Eigenschaften*, Part II, 72., as translated into Italian by Anita Rho [48]. It has not been translated in the present English version in account of the translator's incompetence.]

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[This work has been type-set with ($\mathcal{AMS}$-)LaTeX 2_ε; the calculations in Chap. 3 were made with MATHEMATICA (author's original programs).]